Twitch 166.1

Evan Chen

Twitch Solves ISL

Episode 166

Problem

Let $n \geq 2$ be a positive integer. Prove that any n^2 distinct reals can be placed into a non-singular $n \times n$ matrix.

Solution

We proceed by induction, showing that the result is true even for $n = 1$ as long as the single number is not zero. For $n = 1$ the base case is clear.

So now assume $n \ge 2$. We take any $(n-1)^2$ nonzero entries and commit to some combination of them that makes the lower-right $(n-1) \times (n-1)$ matrix invertible. Take any n of the remaining entries whose sum is nonzero, say $a_1, \dots, a_n$ (we can do this because $n \ge 2$ means for any choice of the first $n-1$ elements at most one choice of a_n will fail). Arbitrary label the remaining entries b_2 through b_n. Then consider matrices of the form

$$M_\pi = \begin{bmatrix} a_{\pi(1)} & a_{\pi(2)} & a_{\pi(3)} & \dots & a_{\pi(n)} \\ b_2 & x_{11} & x_{12} & \dots & x_{1(n-1)} \\ b_3 & x_{21} & x_{22} & \dots & x_{2(n-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_n & x_{(n-1)1} & x_{(n-1)2} & \dots & x_{(n-1)(n-1)} \end{bmatrix}$$

for permutations π on $\{1, \dots, n\}$. Then

$$\det M_\pi = \sum D_i a_{\pi(i)}$$

for some constants D_i, with $D_1 \neq 0$. We contend that at least one choice of the permutations gives a nonzero sum.

Indeed, assume $\det M_\pi$ is the same for every π. Then by considering the change when one swaps the values of $\pi(i)$ and $\pi(j)$ we quickly conclude $D_i = D_j$ for all $i \neq j$. In particular, we should actually have the constant value is

$$\det M_\pi = D_1 \cdot (a_1 + \dots a_n)$$

which is nonzero and we're done.