ELMO SL 2025 G7

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Twitch Solves ISL

Episode 166

Problem

Let ABC be an acute scalene triangle. Points D, E, and F are chosen on $\overline{BC}$, $\overline{AC}$, and $\overline{AB}$, respectively, such that $\angle BDF = \angle CDE$. The circumcircles of triangles BDE and CDF meet line EF again at X and Y, respectively, and they meet each other again at P. Lines BX and CY meet at Q. Show that P, A, and Q are collinear.

External Link

https://aops.com/community/p35216661

Solution

Start by deleting A from the picture. First, note that:

Claim. P, Q, X, Y are cyclic.

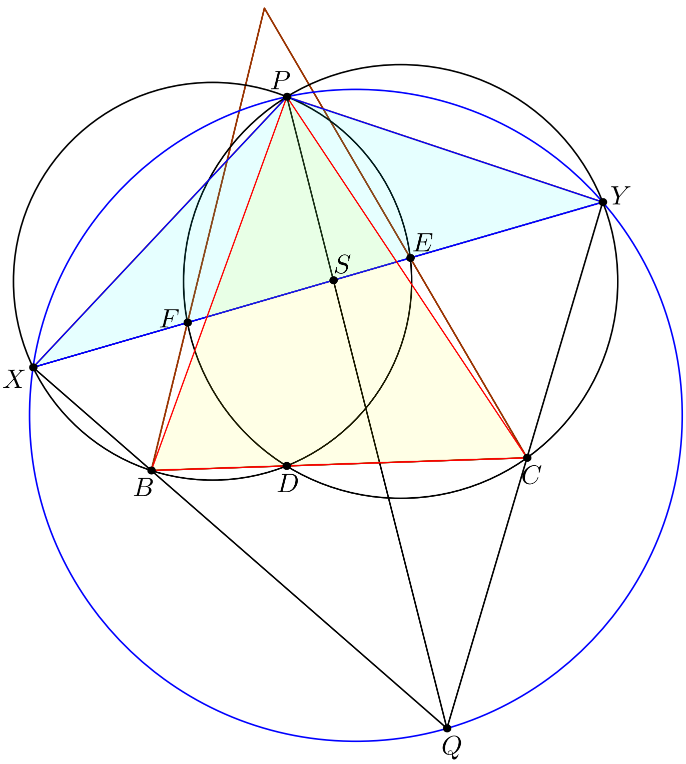
Proof. By Miquel theorem on $\triangle QBC$ with points D, Y, X, on opposite side. □

We note the following angle chase now:

Claim. $\overline{PQ}$ is the angle bisector of $\measuredangle XPY$.

Proof. $\measuredangle XPQ = \measuredangle XYQ = \measuredangle FYC = \measuredangle FDC = \measuredangle FDB$. Similarly, $\measuredangle QPY = \measuredangle CDE$. These were given to be equal. □

Define $S = \overline{XFEY} \cap \overline{PQ}$. Now the problem statement is the same as showing BF, CE, PSQ concurrent.



We use the following lemma to approach this concurrence.

Lemma (Double Menelaus)**.** Lines FB, SQ, EC are concurrent if and only if

$$\frac{SF}{FX} \cdot \frac{XB}{BQ} \cdot \frac{QC}{CY} \cdot \frac{YE}{ES} = 1$$

with directed ratios.

Proof. Use Menelaus on $\triangle SXQ$ and $\triangle QYS$. □

Remark. This double Menelaus lemma is useful in general. It can be phrased in the following more symmetric way: suppose $ABCD$ is any quadrilateral and $P \in \overline{AB}$, $Q \in \overline{BC}$, $R \in \overline{CD}$ and $S \in \overline{DA}$. Then lines PQ, AC, RS are concurrent if and only if $AP/PB \cdot BQ/QC \cdot CR/RD \cdot DS/SA = 1$. In the present problem, the "quadrilateral" is actually degenerate in that three of its vertices are collinear.

However, note the following spiral similarity.

Claim. We have

$$XFS \overset{P}{\mapsto} QCY$$

In particular, $XF/FS = QC/CY$.

Proof. The usual spiral similarity lemma at P (via $\measuredangle XPQ = \measuredangle XYQ = \measuredangle FYC = \measuredangle FPC$) gives us a spiral similarity at P mapping $\overline{XF}$ to $\overline{QC}$. We also know $\measuredangle XPS = \measuredangle XPQ = \measuredangle QPY$, so we get the additional mapping $S \mapsto Y$. □

In the same way $YE/ES = PB/BX$. So the double Menelaus lemma finishes the problem.