# math.SE 5095573

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Twitch Solves ISL

Episode 165

## Problem

Let $n$ be a positive integer. Show that

$$a_1 \cdots a_n + \frac{1}{a_1 \cdots a_n}$$

is a polynomial in $a_1 + \frac{1}{a_2}$, $a_2 + \frac{1}{a_3}$, ..., $a_n + \frac{1}{a_1}$.

## External Link

https://math.stackexchange.com/q/5095573

## Solution

We show two solutions.

**Linear algebra approach.** Define the matrices

$$T_1 = \begin{bmatrix} 0 & 1 \\ -1 & a_1 + \frac{1}{a_2} \end{bmatrix}, \qquad T_2 = \begin{bmatrix} 0 & 1 \\ -1 & a_2 + \frac{1}{a_3} \end{bmatrix}, \qquad T_3 = \begin{bmatrix} 0 & 1 \\ -1 & a_3 + \frac{1}{a_1} \end{bmatrix}, \qquad \dots.$$

Note that $\det T_i = 1$. Note also that

$$T_1 \cdot \begin{bmatrix} a_1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{1}{a_2} \end{bmatrix} = \frac{1}{a_2} \begin{bmatrix} a_1 \\ 1 \end{bmatrix}.$$

So, if we compose the operators in sequence to get a single matrix $T$, we find that

$$T \begin{bmatrix} a_1 \\ 1 \end{bmatrix} = T_n T_{n-1} \dots T_1 \begin{bmatrix} a_1 \\ 1 \end{bmatrix} = \frac{1}{a_1 \dots a_n} \begin{bmatrix} a_1 \\ 1 \end{bmatrix}.$$

However, since $\det T = 1$, we conclude the eigenvalues of $T$ are exactly $a_1 \dots a_n$ and $\frac{1}{a_1 \dots a_n}$. In particular the trace is given by

$$\operatorname{Tr} T = a_1 \dots a_n + \frac{1}{a_1 \dots a_n}.$$

However, every entry of $T$ is a polynomial in $a_i + \frac{1}{a_{i+1}}$ given by matrix multiplication. This proves the result.

**Combinatorial approach (outline).** We are going to give an exact description of the polynomial. Let $X_1 = a_1 + \frac{1}{a_2}$, and so on. Also, let's say a nonempty subset $S \subseteq \{1, \dots, n\}$ is *even-spaced* if when the elements of $S$ are marked in a circle, the gaps between consecutive elements in $S$ are even.

The claim is the following: for some constant $c$,

$$a_1 \cdots a_n + \frac{1}{a_1 \cdots a_n} = c + \sum_{S \text{ even-spaced}} (-1)^{\frac{n-|S|}{2}} \prod_{i \in S} X_i$$

where the sum is taken over all nonempty subsets $S \subseteq \{1, \dots, n\}$ with the property that For example, for $n = 7$ it would equal

$$\begin{aligned} c &+ X_1X_2X_3X_4X_5X_6X_7 \\ &- \sum_{\text{cyc}} X_1X_2X_3X_4X_5 \\ &+ \sum_{\text{cyc}} X_1X_2X_3 + \sum_{\text{cyc}} X_1X_2X_5 \\ &- \sum_{\text{cyc}} X_1. \end{aligned}$$

With this formula written down, the proof essentially follows by principle inclusion-exclusion.

It may be easier to see from an example than by writing notation. Consider for $n = 7$ the sum above and the term $\frac{a_1 a_2}{a_6}$. This term can only arise from an $S$ with $\{1, 2, 5\} \subseteq S$, since these are the only sources of $a_1$, $a_2$, and $\frac{1}{a_6}$. In fact, it comes from the following four terms exactly:

$$\begin{aligned} X_1X_2X_3X_4X_5X_6X_7 &\implies +a_1 \cdot a_2 \cdot \left(a_4 \cdot \frac{1}{a_4}\right) \cdot \frac{1}{a_6} \cdot \left(\frac{1}{a_7} \cdot a_7\right) \\ X_1X_2X_5X_6X_7 &\implies -a_1 \cdot a_2 \cdot \frac{1}{a_6} \cdot \left(\frac{1}{a_7} \cdot a_7\right) \\ X_1X_2X_3X_4X_5 &\implies -a_1 \cdot a_2 \cdot \left(a_4 \cdot \frac{1}{a_4}\right) \cdot \frac{1}{a_6} \\ X_1X_2X_5 &\implies +a_1 \cdot a_2 \cdot \frac{1}{a_6}. \end{aligned}$$

and indeed the total contribution is 0.

(Meanwhile, a term like $\frac{a_1a_2}{a_5}$ or $a_1a_2a_4$ will never appear at all, since $\{1, 2, 4\}$ is not even-spaced. This is despite there being even-spaced subsets containing $\{1, 2, 4\}$. In general, for a nonconstant term to be possible, the set of indices needed for it must itself be even-spaced.)