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Evan Chen

Twitch Solves ISL

Episode 164

Problem

There are $n \geq 3$ positive real numbers $a_1, a_2, \ldots, a_n$. For each $1 \leq i \leq n$ we let $b_i = \frac{a_{i-1}+a_{i+1}}{a_i}$ with indices modulo n. Assume that for all i and j in the range 1 to n, we have $a_i \leq a_j$ if and only if $b_i \leq b_j$. Prove that $a_1 = a_2 = \cdots = a_n$.

External Link

https://aops.com/community/p27522955

Solution

Claim. We have $\max b_i \le 2$.

Proof. Suppose $M = \operatorname{argmax}_i a_i$ (i.e. that $a_M = \max a_i$). Since the a_i's and b_i's have the same ordering, then we should also have

$$b_M = \max b_i.$$

But

$$b_M = \frac{a_{M-1} + a_{M+1}}{a_M} \le 2.$$

□

For the same reason we conclude

$$\min b_i \ge 2.$$

So all the b_i's must be equal to 2, hence all a_i's are equal.