Canada 2025/2

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Twitch Solves ISL

Episode 164

Problem

Determine all positive integers a, b, c, p where p and $p+2$ are odd primes and

$$2^a p^b = (p+2)^c - 1.$$

External Link

https://aops.com/community/p34216137

Solution

The answer is only $(a, b, c, p) = (3, 1, 2, 3)$ which works.

Let $q := p + 2$. Note that the left-hand side contains a factor of $(p + 2) - 1 = p + 1$. Since $\gcd(p, p + 1) = 1$, it follows $p + 1 = q - 1$ must be a power of 2, say 2^n.

However, $2^n - 1$ can be prime only when n is prime while $2^n + 1$ can be prime only when n is a power of 2. So this forces $n = 2$, i.e. $p = 3$ and $q = 5$.

Hence, the problem reduces to solving $2^a \cdot 3^b = 5^c - 1$. Zsigmondy's theorem now ensures there are no solutions for $c \geq 2$, and when $c = 1$ we get $b = 0$ which is invalid.