EGMO 2025/4

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Twitch Solves ISL

Episode 163

Problem

Let ABC be an acute triangle with incenter I and $AB \neq AC$. Let lines BI and CI intersect the circumcircle of ABC at $P \neq B$ and $Q \neq C$, respectively. Consider points R and S such that $AQRB$ and $ACSP$ are parallelograms. Let T be the point of intersection of lines RB and SC. Prove that points R, S, T, and I are concyclic.

Video

https://youtu.be/Nu_RjYVYHPw

External Link

https://aops.com/community/p34542424

Solution

First, we get rid of point T by noting that

$$\measuredangle RTS = \measuredangle(RT, ST) = \measuredangle(QA, PA) = \measuredangle QAP.$$

Hence, it suffices to calculate $\measuredangle RIS$. However, using the vector identity

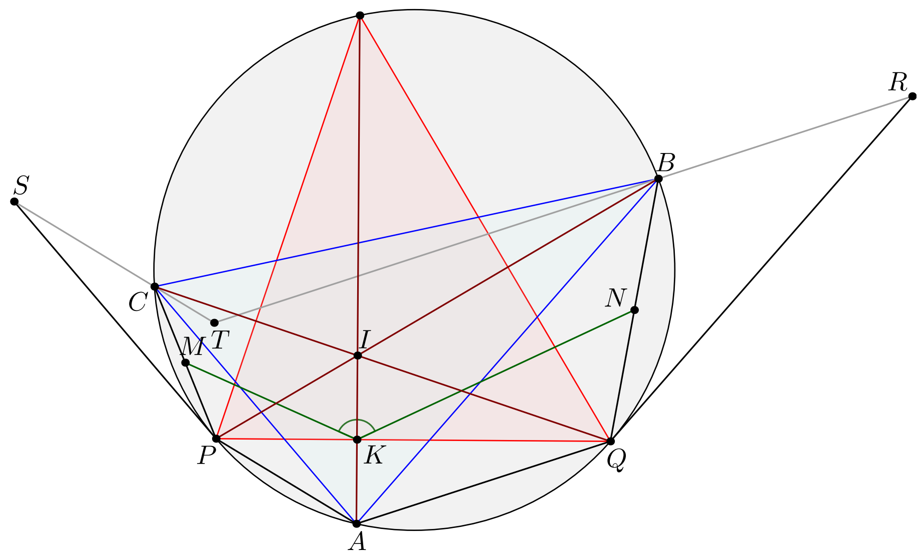
$$\begin{aligned} \vec{R} &= \vec{B} + \vec{Q} - \vec{A} \\ \vec{S} &= \vec{C} + \vec{P} - \vec{A} \end{aligned}$$

we can conclude that

$$\measuredangle(\vec{R} - \vec{I}, \vec{S} - \vec{I}) = \measuredangle(\vec{B} + \vec{Q} - \vec{I} - \vec{A}, \vec{C} + \vec{P} - \vec{I} - \vec{A}) = \measuredangle NKM$$

where $K = \overline{PQ} \cap \overline{AI}$ is the midpoint of $\overline{IA}$, and M and N the midpoints of $\overline{PC}$ and $\overline{BQ}$. Hence, it suffices to prove:

$$\measuredangle QAP = \measuredangle NKM.$$



In fact, we have:

Claim. $\triangle BIC \overset{+}{\sim} \triangle QAP$.

Proof. Because $\measuredangle CBI = \measuredangle IBA = \measuredangle PBA = \measuredangle PQA$ and similarly $\measuredangle BCI = \measuredangle QPA$. $\square$

Hence by the *mean geometry theorem* we in fact have the similarity

$$\triangle BIC \overset{+}{\sim} \triangle NKM \overset{+}{\sim} \triangle QAP$$

since each vertex of $\triangle NKM$ is the midpoint of the two corresponding vertices of $\triangle BIC \sim \triangle QA$. The resulting angle similarity is then immediate.