EGMO 2025/2

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Twitch Solves ISL

Episode 163

Problem

An infinite increasing sequence $a_1 < a_2 < a_3 < \cdots$ of positive integers is called *central* if for every positive integer n, the arithmetic mean of the first a_n terms of the sequence is equal to a_n.

Show that there exists an infinite sequence b_1, b_2, b_3, $\ldots$ of positive integers such that for every central sequence $(a_n)_n$, there are infinitely many positive integers n with $a_n = b_n$.

Video

https://youtu.be/qm5tzRWuQpI

External Link

https://aops.com/community/p34560823

Solution

We choose $b_n = 2n - 1$.

Fix a central sequence. We say an integer N *appears* if $a_i = N$ for some i; this implies

$$a_1 + \cdots + a_N = N^2$$

and we use freely that arbitrarily large integers appear. We define the sequence of *gaps* $(a_{i+1} - a_i)_{i\geq 1}$; if $g = a_{i+1} - a_i$, we say a gap of g occurs at index i.

We perform casework on whether 1 appears infinitely often among the gaps.

Case where infinitely many gaps equal 1. Suppose there are infinitely many integers N such that $N - 1$ and N both appear. Then

$$a_N = (a_1 + \cdots + a_N) - (a_1 + \cdots + a_{N-1})^2 = N^2 - (N-1)^2 = 2N - 1$$

so each such integer N works.

Case where finitely many gaps equal 1. In the rest of the solution we let L denote the number of gaps of 1.

Claim. There are also at most L gaps greater than 2. In other words, there exists constants k and n_0 such that we have $a_n = 2n + k$ for all $n > n_0$.

Proof. Consider any integer N which appears and is large enough that all gaps of 1 occur before index N. Choose another integer C such that $N + C$ appears; then

$$\begin{aligned}
(N+C)^2 &= a_1 + \cdots + a_{N+C} \\
&= N^2 + a_{N+1} + a_{N+2} + \cdots + a_{N+C} \\
&\geq N^2 + (a_N + 2) + (a_N + 4) + \cdots + (a_N + 2C) \\
&= N^2 + C \cdot a_N + (2 + \cdots + 2C) \\
&= N^2 + C \cdot A_N + C(C+1) \\
&= N^2 + (a_N + 1) \cdot C + C^2
\end{aligned}$$

Comparing these, we find that $a_N \leq 2N - 1$.

Now assume for contradiction there were more than L gaps of length 2. If we further pick N large enough that at least $L + 1$ gaps of length 2 occur before index N, then the sum of the gaps up to N is large enough to give $a_N > 2N - 1$, which is a contradiction. $\square$

In particular, if $n > n_0$ is large enough that $2n + k > n_0$ then both $2n + k + 2$ and $2n + k$ appear and

$$\begin{aligned}
(2n+k+2)^2 - (2n+k)^2 &= (1 + \cdots + a_{2n+k+2}) - (1 + \cdots + a_{2n+k}) \\
&= a_{2n+k+2} + a_{2n+k+1} \\
&= 2(2n+k+2) + k + 2(2n+k+1) + k \\
\implies 4(2n+k) + 4 &= (4n + 2k + 4) + k + (4n + 2k + 2) + k \\
\implies k &= -1
\end{aligned}$$

so in fact $a_n = 2n - 1$ for almost all integers n in this case.