EGMO 2025/1

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Twitch Solves ISL

Episode 163

Problem

For a positive integer N, let $c_1 < c_2 < \cdots < c_m$ be all positive integers smaller than N that are coprime to N. Find all $N \geqslant 3$ such that

$$\gcd(N, c_i + c_{i+1}) \neq 1$$

for all $1 \leq i \leq m - 1$.

Video

https://youtu.be/2g3avY_TJAU

External Link

https://aops.com/community/p34560781

Solution

The answer is N even and N powers of 3.

Begin by noting

- Even N all work: all c_i are odd, so each $c_i + c_{i+1}$ work.
- Odd N not divisible by 3 fail: note that $c_1 = 1$ and $c_2 = 2$.

Hence, the problem reduces checking that if N is an *odd* multiple of 3, then it works if and only if N is a power of 3.

If N was indeed a power of 3, then $(c_i)_i = (1, 2, 4, 5, 7, 8, \dots)$ consists of all the non-multiples of 3 in order. In this case, each $c_i + c_{i+1}$ is necessarily $0 \pmod 3$, so these numbers do work.

The final case:

Claim. Suppose $N = 3^e d$, where $e \geq 1$, and $d > 1$ is not a multiple of 3 (and odd). Then

- If $d \equiv 1 \pmod 6$, then $d-2$ and $d+1$ are consecutive c_i's whose sum $2d-1$ is coprime to N.
- If $d \equiv 5 \pmod 6$, then $d-1$ and $d+2$ are consecutive c_i's whose sum $2d+1$ is coprime to N.

Proof. We prove just the first bullet; the second calculation looks exactly the same. Assuming $d \equiv 1 \pmod 6$, Note that

$$\begin{aligned} 3 &\mid \gcd(d-1, N) \\ d &= \gcd(d, N) \\ \gcd(d-2, N) &= \gcd(d-2, 3^e d) = \gcd(d-2, 3^e \cdot 2) = 1 \\ \gcd(d+1, N) &= \gcd(d+1, 3^e d) = \gcd(d+1, -3^e) = 1. \end{aligned}$$

Moreover,

$$\gcd(2d-1, N) = \gcd(2d-1, 3^e d) = \gcd(2d-1, 3^e \cdot 2d) = \gcd(2d-1, 3^e) = 1.$$

This completes the proof. $\square$