Twitch 161.2

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Twitch Solves ISL

Episode 162

Problem

Let $ABCDEF$ be a convex hexagon with $AB = DE$, $CD = FA$, $EF = BC$ and

$$\angle A + \angle C + \angle E = \angle B + \angle D + \angle F.$$

Show that triangles ACE and BDF have the same area.

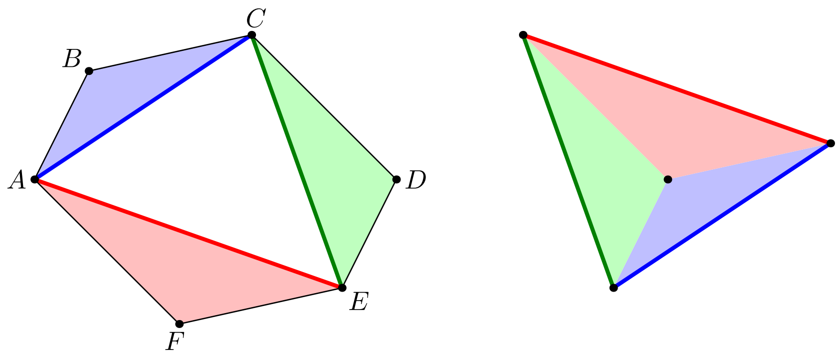
Video

https://youtu.be/bUi8M1DWseQ

Solution

Rewrite the angle condition as $\angle A + \angle C + \angle E = 360^\circ$. We show two approaches.

Slick solution. Dissect $ABCDEF$ by cutting it into triangles ABC, CDE, EFA, and ACE. Then the problem condition ensures that ABC, CDE, EFA can be rearranged to glue together to another triangle congruent to ACE. Hence, the area of ACE is half the total hexagon.



Since the same is true for BDE, we're done: $[ACE] = [BDF] = \frac{1}{2}[ABCDEF]$.

Second normie solution. We will show that in fact, the opposite sides of the hexagon are parallel too. In this way, the hexagon is uniquely determined by the points A, B, C, D (as E and F will be reflections of B and C through the midpoint of AD).

Treat quadrilateral $ABCD$ as fixed, let $\alpha = \angle A$ vary. Then

- As a function of α, the length DF is monotonic in α (because $DF^2 = AF^2 + AD^2 - 2 \cdot AD \cdot AF \cos(\alpha - \angle BAD)$).
- Similarly, the angle $\angle DEF$ is monotonic in the length of DF, by a similar law of cosines.
- Hence for a given α, the angle $\angle DEF$ is monotonic in α.
- Since $\alpha + \angle DEF$ is fixed, there can be at most one valid α.
- Choosing the special hexagon described above establishes the claim.

In particular triangles ACE and BDF are manifestly congruent, so in particular they're congruent.