HMIC 2025/3

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Twitch Solves ISL

Episode 162

Problem

Let $ABCD$ be a parallelogram, and let O be a point inside $ABCD$. Suppose the circumcircles of triangles OAB and OCD intersect at $P \neq O$, and the circumcircles of triangles OBC and OAD intersect at $Q \neq O$. Prove $\angle POQ$ equals one of the angles of quadrilateral $ABCD$.

Video

https://youtu.be/vhC62b_GHgs

Solution

Let M be the center of the parallelogram. We start with the following critical observation:

Claim. P and Q are also reflections about M.

Proof. Let P' be the reflection of P across M. We need to check that P' lies on (BOC) and (AOD). The former follows from

$$\begin{aligned}\measuredangle BP'C &= \measuredangle DPA = \measuredangle DPO + \measuredangle OPA \\ &= \measuredangle DCO + \measuredangle OBA = \measuredangle(\overline{DC}, \overline{OC}) + \measuredangle(\overline{OB}, \overline{BA}) \\ &= \measuredangle(\overline{DC}, \overline{BA}) + \measuredangle(\overline{OB}, \overline{OC}) = 0 + \measuredangle BOC = \measuredangle BOC.\end{aligned}$$

The latter is symmetric. $\square$

To finish the angle chase, since $APCQ$ is a parallelogram we can write

$$\begin{aligned}\measuredangle QOP &= \measuredangle(\overline{QO}, \overline{QC}) + \measuredangle(\overline{QC}, \overline{PA}) + \measuredangle(\overline{PA}, \overline{PO}) \\ &= \measuredangle OQC + 0 + \measuredangle APO \\ &= \measuredangle OBC + \measuredangle ABO = \measuredangle ABC\end{aligned}$$

as needed.