EGMO 2025/5

Evan Chen

Twitch Solves ISL

Episode 162

Problem

Fix an integer $n > 1$. In a configuration of an $n \times n$ board, each of the n^2 cells contains an arrow, either pointing up, down, left, or right. Given a starting configuration, Turbo the snail starts in one of the cells of the board and travels from cell to cell. In each move, Turbo moves one square unit in the direction indicated by the arrow in her cell (possibly leaving the board). After each move, the arrows in all of the cells rotate $90°$ counterclockwise. We call a cell *good* if, starting from that cell, Turbo visits each cell of the board exactly once, without leaving the board, and returns to her initial cell at the end.

Determine, in terms of n, the maximum number of good cells over all possible starting configurations.

Video

https://youtu.be/GbAvWnsCSZk

External Link

https://aops.com/community/p34542428

Solution

Odd n. When $n > 1$ is odd, there's no way to navigate loop through the board at all visiting every cell exactly once, and hence good cells cannot exist at all. The answer is 0 in that case.

Even n. For even n, we claim that the answer is $n^2/4$. In fact, we prove something more strong: if there are any good cells at all, then there are exactly $n^2/4$ good cells.

We assume henceforth that there is some good cell and hence that there is at least one valid loop (and indeed for even n there are valid directed Hamiltonian cycles of an $n \times n$ board so this case can occur). The number of steps in the loop is n^2 and hence divisible by 4: thus it's easy to see that that every fourth cell along the loop is also good, because the same loop will be traced out. This gives $n^2/4$ good cells.

To prove that there can't be more good cells, it's sufficient to focus on the northwest corner.

Claim. There are only four methods for Turbo to pass through that corner, which we illustrate below. The subscript number indicates the order the cells are visited in relative to each other.

$$\begin{bmatrix} \downarrow_2 & \uparrow_3 \\ \uparrow_1 & \end{bmatrix} \qquad \begin{bmatrix} \downarrow_2 & \leftarrow_3 \\ \uparrow_1 & \end{bmatrix} \qquad \begin{bmatrix} \leftarrow_2 & \leftarrow_1 \\ \leftarrow_3 & \end{bmatrix} \qquad \begin{bmatrix} \leftarrow_2 & \leftarrow_1 \\ \uparrow_3 & \end{bmatrix}$$

Proof. Clear. □

From now on, we say two boards are *rotations* if one can be obtained from the other by rotating all the arrows the same amount (i.e. each board has four possible rotations).

The final observation is that in the four methods in the claim, none of the boards are rotations of each other. Hence for a given starting board, given a Hamiltonian cycle, the time modulo 4 that Turbo passes through the northwest corner is uniquely determined. Then, by simply tracing out Turbo's trajectory from there, the entire directed Hamiltonian cycle is uniquely determined too, i.e. for every cell we know exactly which cell Turbo goes to next and the time modulo 4 that Turbo visited the cell. This shows the good cells we asserted above are the only ones.