EGMO 2025/3

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Twitch Solves ISL

Episode 161

Problem

Let ABC be an acute triangle. Points B, D, E, and C lie on a line in this order and satisfy $BD = DE = EC$. Let M and N be the midpoints of AD and AE, respectively. Suppose triangle ADE is acute, and let H be its orthocenter. Points P and Q lie on lines BM and CN, respectively, such that D, H, M, and P are concyclic and pairwise different, and E, H, N, and Q are concyclic and pairwise different. Prove that P, Q, N, and M are concyclic.

Video

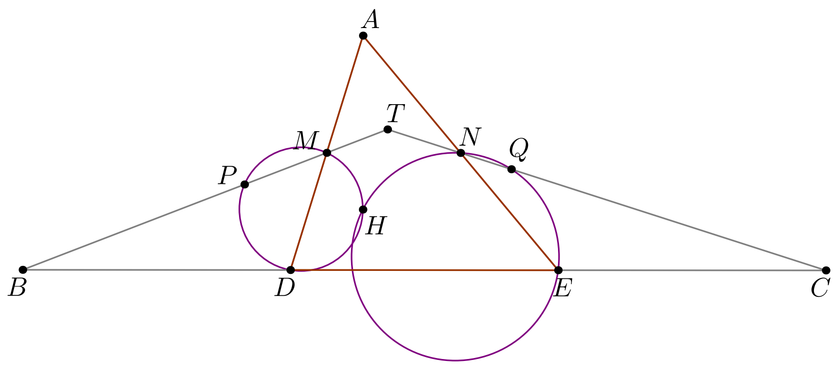
https://youtu.be/yiRQt-OhbtA

External Link

https://aops.com/community/p34533363

Solution

We proceed by barycentric coordinates on $\triangle ADE$. Let $a = DE$, $b = EA$, $c = AD$. Recall $H = (S_B S_C : S_C S_A : S_A S_B)$. Finally, let T be the intersection of lines BM and CN.



Claim. We have $T = (3 : 1 : 1)$.

Proof. Write $B = (0 : 2 : -1)$ and $M = (1 : 1 : 0)$. Note that

$$\det \begin{bmatrix} 3 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 2 & -1 \end{bmatrix}$$

so T lies on BM. □

Claim. Point T lies on the radical axis of (DMH) and (ENH).

Proof. The circumcircle of (DMH) then has equation given by

$$0 = -a^2yz - b^2zx - c^2xy + (x+y+z)\left(\frac{c^2}{2}x + wz\right)$$

for some constant w (by plugging in D and H). To determine w, plug in H to get

$$\begin{aligned} \frac{c^2}{2}S_BS_C + w \cdot S_AS_B &= S_AS_BS_C \cdot \frac{a^2S_A + b^2S_B + c^2S_C}{S_BS_C + S_CS_AS_AS_B} = S_AS_BS_C \cdot \frac{8[ABC]^2}{4[ABC]^2} \\ \implies \frac{c^2}{2}S_C + S_A \cdot w &= 2S_AS_C \\ \implies S_A \cdot w &= S_C\left(b^2 + \frac{1}{2}c^2 - a^2\right). \end{aligned}$$

In other words, (DMH) has equation given by

$$0 = -a^2yz - b^2zx - c^2xy + (x+y+z)\left(\frac{c^2}{2}x + \frac{S_C(b^2 + c^2/2 - a^2)}{S_A}z\right).$$

Similarly, (CNH) has equation given by

$$0 = -a^2yz - b^2zx - c^2xy + (x+y+z)\left(\frac{b^2}{2}x + \frac{S_B(c^2 + b^2/2 - a^2)}{S_A}y\right).$$

To check $T = (3:1:1)$ lies on the radical axis, it suffices to check equality of the linear parts, that is:

$$\frac{c^2}{2} \cdot 3 + \frac{S_C(b^2 + c^2/2 - a^2)}{S_A} = \frac{b^2}{2} \cdot 3 + \frac{S_B(c^2 + b^2/2 - a^2)}{S_A}.$$

Using a common denominator for the left-hand side, we get

$$\begin{aligned} \frac{c^2}{2} \cdot 3 + \frac{S_C(b^2 + c^2/2 - a^2)}{S_A} &= \frac{3c^2 S_A + S_C(2b^2 + c^2 - 2a^2)}{2S_A} \\ &= \frac{3c^2(b^2 + c^2 - a^2) + (a^2 + b^2 - c^2)(2b^2 + c^2 - 2a^2)}{2S_A} \\ &= \frac{b^4 + b^2c^2 + c^4 - a^4}{S_A}. \end{aligned}$$

□

Since this is symmetric in b and c, we're done.

Since T lies on the radical axis, it follows that $TM \cdot TP = TN \cdot TQ$ and the problem is solved.