OMF 2025/4

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Twitch Solves ISL

Episode 159

Problem

Determine all bounded sequences of positive integers a_1, a_2, $\ldots$ such that for any prime number p and for any index $n \geq 1$, the number $a_n a_{n+1} \ldots a_{n+p-1} - a_{n+p}$ is a multiple of p.

Video

https://youtu.be/PilRVBNcWxY

Solution

Only constant sequences can work, and by Fermat's little theorem they all work.

Let $M = \max a_i$.

Claim 1. The sequence is periodic.

Proof. Take $p > M$. The residues modulo p of any p consecutive terms determine the next term, and there are only p^p possibilities. So it's eventually periodic modulo p, but $p > M$, so one can work backwards and it's actually periodic. □

Let T be the period of the sequence. Again consider arbitrary $p > 2\max(M, T)$ with $p \equiv -1 \pmod{T}$. Then note

$$\begin{aligned} a_n a_{n+1} \dots a_{n+p-1} &\equiv a_{n+p} \pmod{p} \\ a_{n+1} a_{n+2} \dots a_{n+p} &\equiv a_{n+p+1} \pmod{p} \\ \implies \frac{a_{n+p}^2}{a_n} &\equiv a_{n+p+1} \pmod{p} \end{aligned}$$

Taking indices modulo T gives

$$a_{n+1}^2 \equiv a_n^2 \pmod{p}$$

but since $p > 2M$, we get $a_n = a_{n+1}$, so the sequence is constant.

Remark. Without the condition that a_n is bounded, there are many more examples, like $a_n = cn + c'$ for any constants c and c'.