Twitch 158.2

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Twitch Solves ISL

Episode 158

Problem

Let ABC be a triangle with circumcircle Γ and incircle γ. Let γ_a, γ_b, γ_c be the reflections of γ over the sides BC, CA, AB. Let $\mathcal{R}$ be the region of points which are contained inside Γ and also contained in at least one of γ_a, γ_b, γ_c. [TODO: diagram] Prove that the area of $\mathcal{R}$ is at least the area of γ.

Video

https://youtu.be/LfIOXTYSFGA

Solution

Let P be a point inside γ (or really any point inside ABC). We will show the reflection of P across at least one of the sides lies inside Γ; if so that will solve the problem.

If the triangle is non-acute, say with $\angle A \geq 90^\circ$, then the reflection of P across BC always lies inside Γ.

Otherwise, assume for contradiction all three reflections P lie outside. Then

$$\begin{aligned} \angle BPC &> 2\angle A \\ \angle CPA &> 2\angle B \\ \angle APB &> 2\angle C. \end{aligned}$$

Adding these gives a contradiction.