JMO 2025/4

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Twitch Solves ISL

Episode 158

Problem

Let n be a positive integer, and let $a_0 \geq a_1 \geq \cdots \geq a_n \geq 0$ be integers. Prove that

$$\sum_{i=0}^{n} i \binom{a_i}{2} \leq \frac{1}{2} \binom{a_0 + a_1 + \cdots + a_n}{2}.$$

Video

https://youtu.be/EoTMC4f70QY

External Link

https://aops.com/community/p34335897

Solution

For $n = 0$ (which we permit) there is nothing to prove. Hence to prove by induction on n, it would be sufficient to verify

$$2n\binom{a_n}{2} \le \binom{a_0 + a_1 + \cdots + a_n}{2} - \binom{a_0 + a_1 + \cdots + a_{n-1}}{2}.$$

Rearranging the terms around, that's equivalent to proving

$$\begin{aligned} \iff 2n(a_n^2 - a_n) &\le a_n^2 + a_n \cdot (2(a_0 + \cdots + a_{n-1}) - 1) \\ \iff 0 &\le 2a_n(a_0 + \cdots + a_{n-1} - na_n) + a_n(a_n + 2n - 1). \end{aligned}$$

However, the last line is obvious because $\min(a_0, \dots, a_{n-1}) \ge a_n$, and $a_n \ge 0$.

Remark. The only equality case is when $a_0 \in \{0, 1\}$ and $a_i = 0$ for $i \ge 1$.

The bound in the problem is extremely loose and pretty much anything will work.