XOOK 2025/2

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Twitch Solves ISL

Episode 157

Problem

Two points A and B are placed inside a convex polygonal room $\mathcal{P}$ uniformly at random. Suppose that the expected value of AB^2 is 1. Alice, at point A, opens her flashlight pointing directly opposite B, and lights up an angle of $\alpha < \frac{\pi}{2}$. Find the smallest constant c such that over all possible $\mathcal{P}$ and α, the expected value of the area lit is at most $c \tan \alpha$.

Video

https://youtu.be/kPRQ11q--gY

External Link

https://aops.com/community/p33959008

Solution

The main idea for bounding is the following: the point C lies in the flashlight zone if and only if

$$\angle CAB > 180^\circ - \frac{\alpha}{2}.$$

We prove:

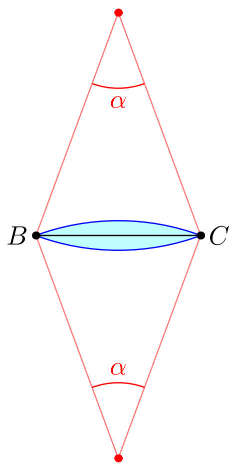
Claim.

$$\mathbb{P}_{A,B,C\in\mathcal{P}}\left[\angle CAB > 180^\circ - \frac{\alpha}{2}\right] = \frac{k(\alpha)\mathbb{E}[BC^2]}{\text{Area}(\mathcal{P})}$$

where

$$k(\alpha) = 2\cdot\frac{\frac{1}{2}\alpha - \frac{1}{2}\sin\alpha}{(2\sin\frac{1}{2}\alpha)^2}.$$

Proof. Fix B and C. Consider the area of points A that work; it's contained inside the "lens" shape sketched below.



The area of this lens shape is given by

$$BC^2\cdot k(\alpha)$$

for some constant $k(\alpha)$ that depends only on α; to be precise, $k(\alpha)$ is the ratio of the area of the lens to the square of its diameter. Explicitly, an annoying calculation gives the formula for $k(\alpha)$ above. □

So the expected value of the lit area is always at most $k(\alpha)\cdot\mathbb{E}[BC^2] = k(\alpha)$. To produce a preliminary bound, we do the following annoying-as-hell calculation:

Claim. The function $\alpha \mapsto \frac{k(\alpha)}{\tan\alpha}$ decreases monotonically for $0 < \alpha < \pi/2$. Moreover, it has limit

$$\lim_{\alpha\to 0^+}\frac{k(\alpha)}{\tan\alpha} = \frac{1}{6}.$$

Proof. The proof of monotonicity is super annoying and omitted. The limit follows from using either L'Hopital or WolframAlpha. □

This gives a proof that $c = 1/6$ works. However, for any fixed polygon $\mathcal{P}$, as $\alpha \to 0$, this bound is sharp, because the probability over (B, C) that the lens actually protrudes out of the polygon approaches 0 as well (here we use the convexity of the polygon).