XOOK 2025/1

Evan Chen

Twitch Solves ISL

Episode 156

Problem

For the Exeter disco party, Oron wants to tile a 2024×2024 square in the middle of the dance floor with 1×1 tiles which have arrows pointing either up, left, right, or down. After placing down all the tiles in some configuration, he realizes that he wants to rotate some of the tiles so that no matter which tile a dancer starts on, after following the arrows they end up leaving the grid through some fixed cell. What's the minimum number of tiles he needs to rotate so that this is always possible?

Video

https://youtu.be/LS4DsHGIp2o

External Link

https://aops.com/community/p33958997

Solution

For a general even number n, the answer is $\frac{(n-2)^2}{2} + 4 \cdot (n-1) - 1$.

Construction. For $n = 8$, consider the following grid (which generalizes obviously). The arrows on the border of the grid are colored red and drawn in "double" (for accessibility).

$$\begin{bmatrix} \Uparrow & \Uparrow & \Uparrow & \Uparrow & \Uparrow & \Uparrow & \Uparrow & \Rightarrow \\ \Leftarrow & \rightarrow & \leftarrow & \rightarrow & \leftarrow & \rightarrow & \leftarrow & \Rightarrow \\ \Leftarrow & \rightarrow & \leftarrow & \rightarrow & \leftarrow & \rightarrow & \leftarrow & \Rightarrow \\ \Leftarrow & \rightarrow & \leftarrow & \rightarrow & \leftarrow & \rightarrow & \leftarrow & \Rightarrow \\ \Leftarrow & \rightarrow & \leftarrow & \rightarrow & \leftarrow & \rightarrow & \leftarrow & \Rightarrow \\ \Leftarrow & \rightarrow & \leftarrow & \rightarrow & \leftarrow & \rightarrow & \leftarrow & \Rightarrow \\ \Leftarrow & \rightarrow & \leftarrow & \rightarrow & \leftarrow & \rightarrow & \leftarrow & \Rightarrow \\ \Downarrow & \Downarrow & \Downarrow & \Downarrow & \Downarrow & \Downarrow & \Downarrow & \Rightarrow \end{bmatrix}$$

It's easy to see that all but one of the red doubled arrows needs to change; in other words at least $4n - 5$ of the red doubled arrows needs to change. Moreover, in each pair of arrows inside facing each other, at least one of the two arrows needs to change as well. This gives the other $\frac{(n-2)^2}{2}$ term.

Algorithm. We now prove this number of steps is always possible by specifying a procedure. Let's refer to the *outer ring* as the outside $4n - 4$ cells (doubled red above) and the *inner square* as the inside $(n-2) \times (n-2)$ cells (single black above).

The algorithm consists of several steps.

Claim. By changing at most half the arrows in the inner square, we can ensure that any dancer in the inner square exits it.

Proof. Count the number of left and right arrows inside the grid, and take whichever is fewer (ties broken arbitrarily) and turn them all the other way. Do the same for up and down arrows.

Then, for example if all arrows are left or up, it's clear there can be no cycles. $\square$

Now for the outer ring of $4n - 4$, we consider two cases.

- Suppose any corner A points outward. Then we can change all the *other* red arrows besides A to point counterclockwise, so they form a "conveyor belt" towards A, as shown below.

$$\begin{bmatrix} \overset{A}{\Uparrow} & \Leftarrow & \Leftarrow & \Leftarrow & \Leftarrow & \Leftarrow & \Leftarrow & \Leftarrow \\ \Downarrow & & & & & & & \Uparrow \\ \Downarrow & & & & & & & \Uparrow \\ \Downarrow & & & & & & & \Uparrow \\ \Downarrow & & & & & & & \Uparrow \\ \Downarrow & & & & & & & \Uparrow \\ \Downarrow & & & & & & & \Uparrow \\ \Rightarrow & \Rightarrow & \Rightarrow & \Rightarrow & \Rightarrow & \Rightarrow & \Rightarrow & \Uparrow \end{bmatrix}$$

- Suppose all the corners point to other red arrows. Choose any corner A and any other corner B. Then we arrange A to be the exit cell and point all the other red arrows in a similar conveyor belt, but choosing the between the two directions (clockwise vs counterclockwise) of the belt so that B does not need to change. (In fact, this can be done in fewer than $4n - 5$ steps always.)