HMMT 2025 T7

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Twitch Solves ISL

Episode 156

Problem

Determine, with proof, whether a square can be dissected into finitely many (not necessarily congruent) triangles, each of which has interior angles 30°, 75°, and 75°.

Video

https://youtu.be/Guwjcgnkq7c

Solution

The answer is no; in fact, no *rectangle* can be so dissected.

Assume for contradiction that such a dissection exists. Then the number of 75° angles in the dissection must be exactly twice the number of 30° angles.

On the other hand, consider a vertex of the dissection. These come in three different kinds:

- The four corners of the square are 90° angles that must be split as

$$90^\circ = 30^\circ + 30^\circ + 30^\circ.$$

- Any 180° angle can be split as either

$$180^\circ = 6 \cdot 30^\circ = 30^\circ + 2 \cdot 75^\circ.$$

- Any 360° angle can be split as

$$360^\circ = 12 \cdot 30^\circ = 7 \cdot 30^\circ + 2 \cdot 75^\circ = 2 \cdot 30^\circ + 4 \cdot 75^\circ.$$

In every case, the number of 75° angles is at most twice the number of 30° angles. Furthermore, equality cannot occur because of the four right angles in the original square. This produces the required contradiction.