H3507300

Evan Chen

Twitch Solves ISL

Episode 156

Problem

Let $\oplus$ denote bitwise XOR. Solve over $\mathbb{Z}_{\geq 0}$ the functional equation

$$f(x \oplus f(y) + y) = (f(x) \oplus y) + y.$$

Video

https://youtu.be/v_Ng9xsa26g

External Link

https://aops.com/community/p34077999

Solution

The answer is f the identity, which obviously works.

Let $P(x, y)$ denote the given statement.

Claim. $f(t) \geq t$ for every t.

Proof. Note $P(f(t), t)$ gives $f(t) = f(f(t)) \oplus t + t \geq t$. $\square$

Claim. $f(0) = 0$.

Proof. Note that

$$P(0, y) \implies y \oplus f(0) + y = f(f(y) + y) \geq f(y) + y \geq 2y.$$

Hence $y \oplus f(0) \geq y$ for all integers $y \geq 0$. This forces $f(0) = 0$ (e.g. by taking $y = f(0)$). $\square$

Now, we find that

$$P(0, y) \implies 2y = f(y + f(y)).$$

If $f(y) > y$ for any y, that RHS is at least $2y$, contradiction.