

Philippines 2024/III/7

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Twitch Solves ISL

Episode 155

Problem

Find integers a and b such that whenever $P(x)$ is a polynomial of degree 3 satisfying $P(2) = 6$, $P(6) = 2$, and $P(1) = P(8)$, then $P(a) = b$.

Video

<https://youtu.be/SZKnEAclM3o>

External Link

<https://aops.com/community/p33312423>

Solution

We claim $P(13) = 72$.

Note that

$$P(x) + x - 8 = \ell(x - 2)(x - 6)(x - k)$$

for some real numbers k and ℓ , since the left-hand side has roots at 2 and 6 and degree 3. Plugging in $x = 1$ and $x = 8$ we get

$$\begin{aligned}P(1) - 7 &= 5\ell(1 - k) \\ P(8) &= 12\ell(8 - k)\end{aligned}$$

so subtraction gives

$$7 = 12\ell(8 - k) - 5\ell(1 - k) = 7\ell(13 - k).$$

Hence, for $x = 13$ we get

$$P(13) + 5 = \ell \cdot 11 \cdot 7 \cdot (13 - k) = 77$$

and so $P(13) = 72$.