ELMO 2024/4

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Episode 151

Problem

Let n be a positive integer. Find the number of sequences $(a_0, a_1, a_2, \dots, a_{2n})$ of integers in $\{0, \dots, n\}$ such that for all integers $0 \le k \le n$ and all integers $m \ge 0$, there exists an integer $k \le i \le 2k$ such that $\lfloor k/2^m \rfloor = a_i$.

Video

https://youtu.be/J4QtlbYxBDU

External Link

https://aops.com/community/p30985799

Solution

The answer is 2^n. We note $a_0 = 0$ and ignore it going forward, focusing only on a_i for $i \geq 1$.

In what follows, for each positive integer t we let

$$D(t) \coloneqq \{t, \lfloor t/2 \rfloor, \lfloor t/4 \rfloor, \dots\}.$$

For example, $D(13) = \{13, 6, 3, 1, 0\}$. Then the problem condition is equivalent to saying that every element of $D(k)$ appears in $\{a_k, \dots, a_{2k}\}$.

We prove the following structure claim about all the valid sequences.

Claim. In any valid sequence, for each $0 \leq k \leq n$,

- a_{2k-1} and a_{2k} are elements of $D(k)$; and
- $a_k, \dots, a_{2k}$ consist of all the numbers from 0 to k each exactly once.

Proof. We proceed by induction; suppose we know it's true for k and want it true for $k+1$. By induction hypothesis:

- $\{a_k, \dots, a_{2k}\}$ contains each of 0 to k exactly once;
- a_k is an element of $D(\lceil k/2 \rceil)$;
- We also know $\{a_{k+1}, \dots, a_{2k+2}\}$ contains all elements of $D(k+1)$ by problem condition.

However, note that

$$D(\lceil k/2 \rceil) \subseteq D(k+1)$$

so that means either

$$(a_{2k+1} = a_k \text{ and } a_{2k+2} = k+1) \quad \text{OR} \quad (a_{2k+1} = k+1 \text{ and } a_{2k+2} = a_k). \qquad \square$$

We return to the problem of counting the sequences. It suffices to show that if $(a_0, \dots, a_{2n})$ is a valid sequence, there are exactly two choices of ordered pairs $(x, y) \in \{0, \dots, n+1\}$ such that $(a_0, \dots, a_{2n}, x, y)$ is a valid sequence. However, the structure claim above implies that $\{x, y\} = \{a_n, n+1\}$, so there are at most two choices. Moreover, both of them work by the structure claim again (because $k = n = 1$ is the only new assertion when augmenting the sequence, and it holds also by the structure claim). This completes the proof.

Remark. Here are some examples to follow along with. When $n = 4$ the 16 possible values of $(a_4, a_5, a_6, a_7, a_8)$ are

$$\begin{array}{cc}
(2,1,3,4,0) & (2,0,3,4,1) \\
(2,1,3,0,4) & (2,0,3,1,4) \\
(0,1,3,4,2) & (1,0,3,4,2) \\
(0,1,3,4,2) & (1,0,3,4,2) \\
(0,1,3,2,4) & (1,0,3,2,4) \\
(2,3,1,4,0) & (2,3,0,4,1) \\
(2,3,1,0,4) & (2,3,0,1,4) \\
(0,3,1,4,2) & (1,3,0,4,2) \\
(0,3,1,2,4) & (1,3,0,2,4)
\end{array}$$

Now the point is that when moving to $n = 5$, the element $a_4 \in \{0, 1, 2\} = D(2) \subseteq D(5)$ is chopped-off, and a_9 and a_{10} must be 5 and the chopped-off element in some order. So each of these sequences extends in exactly two ways, as claimed.