Canada 2024/5

Evan Chen

Twitch Solves ISL

Episode 151

Problem

Initially, three non-collinear points, A, B, and C, are marked on the plane. You have a pencil and a double-edged ruler of width 1. Using them, you may perform the following operations:

- Mark an arbitrary point in the plane.
- Mark an arbitrary point on an already drawn line.
- If two points P_1 and P_2 are marked, draw the line connecting P_1 and P_2.
- If two non-parallel lines ℓ_1 and ℓ_2 are drawn, mark the intersection of ℓ_1 and ℓ_2.
- If a line ℓ is drawn, draw a line parallel to ℓ that is at distance 1 away from ℓ (note that two such lines may be drawn).

Prove that it is possible to mark the orthocenter of ABC using these operations.

Video

https://youtu.be/bu_C5NzK1T4

External Link

https://aops.com/community/p30115331

Solution

We prove the following series of claims.

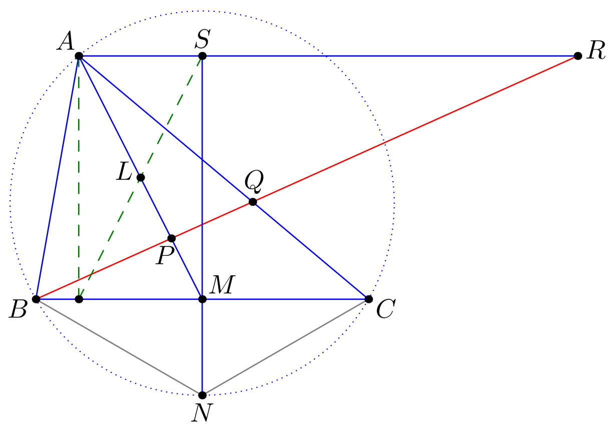
Claim. Given two intersecting lines ℓ_1 and ℓ_2, we get the two angle bisectors.

Proof. Make a parallelogram two of whose sides are ℓ_1 and ℓ_2 and where the distance to opposite sides is 1. This is actually a rhombus. □

Claim. Given a line segment $\overline{XY}$ we can find its midpoint.

Proof. Draw two lines parallel to $\overline{XY}$ distance 1 away, say ℓ and ℓ'. Let Z be on ℓ'. Let rays ZX and ZY meet ℓ again and X' and Y'. Then we can get the centroid of $\triangle X'Y'Z$. □

From now on fix triangle ABC. Our goal is to draw the A-altitude; see the figure below.



Claim. We can construct the perpendicular bisectors of side $\overline{BC}$, which meets $\overline{BC}$ at its midpoint M.

Proof. Bisect all the angles of ABC to get the incenter I and excenters I_A. Let N be the midpoint of $\overline{II_A}$ which is equidistant from B and C. Then the internal bisector of $\angle BNC$ is the perpendicular bisector of side BC. □

To finish, an arbitrary line g through B inside the triangle, and let it meet $\overline{AM}$ at P and $\overline{AC}$ at Q. Using a straightedge alone we can construct the harmonic conjugate R of P with respect to $\overline{BQ}$. Then from $(BC; M\infty) = -1$, where ∞ is the point at infinity along line BC, we conclude that $\overline{AR} \parallel \overline{BC}$.

Bisect $\overline{AM}$ to get L. Let line AR meet the perpendicular bisector of BC at S. Then $\overline{SL}$ passes through the foot of the A-altitude and we're done.