Philippines 2024/1

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Twitch Solves ISL

Episode 150

Problem

Let $f\colon \mathbb{Z}^2 \to \mathbb{Z}$ be a function satisfying

$$f(x+1,y)+f(x,y+1)+1=f(x,y)+f(x+1,y+1)$$

for all integers x and y. Can it happen that $|f(x,y)| \le 2024$ for all $x, y \in \mathbb{Z}$?

Video

https://youtu.be/pm4lAI17Hxc

External Link

https://aops.com/community/p29991351

Solution

No, it can't happen.

Observing that $f(x, y) = xy$ happens to work, let $g(x, y) = f(x, y) - xy$. Then the given condition on f implies that

$$g(x+1, y) + g(x, y+1) = g(x, y) + g(x+1, y+1).$$

It turns out we can classify all the functions g satisfying this equation.

Claim. If $g(x, y)$ satisfies the above equation, then in fact

$$g(x, y) = g(x, 0) + g(0, y) - g(0, 0).$$

Proof. For $x, y \geq 0$ it follows easily by induction on $|x| + |y|$. The proof for the other quadrants is the same. □

Translating back, this implies

$$f(x, y) = xy + f(x, 0) + f(0, y) - f(0, 0).$$

So it's obvious f can't be bounded by 2024.

Remark. In fact, this basically classifies all the functions f.