EGMO 2022/1

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Twitch Solves ISL

Episode 149

Problem

Let ABC be an acute-angled triangle in which $BC < AB$ and $BC < CA$. Let point P lie on segment AB and point Q lie on segment AC such that $P \neq B$, $Q \neq C$ and $BQ = BC = CP$. Let T be the circumcenter of triangle APQ, H the orthocenter of triangle ABC, and S the point of intersection of the lines BQ and CP. Prove that T, H, and S are collinear.

Video

https://youtu.be/tjxbdw41fzc

External Link

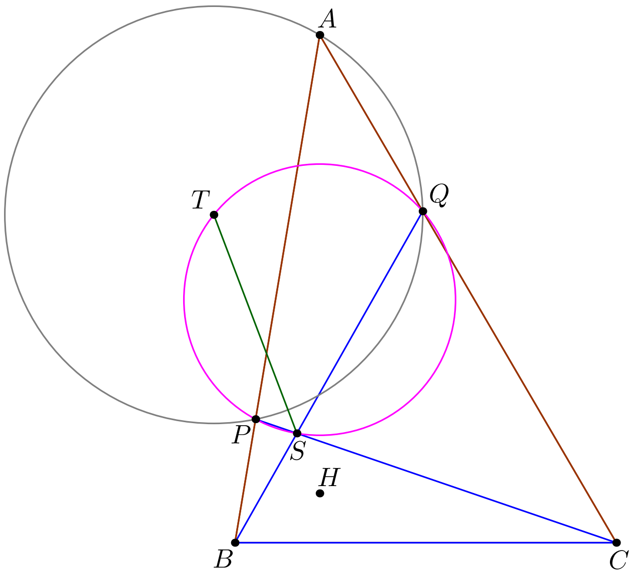
https://aops.com/community/p24921857

Solution

We start by eliminating H:

Claim. The point H is the incenter of $\triangle SBC$.

Proof. Note that because $\overline{BH}$ is the angle bisector of $\angle SBC$ in isosceles triangle BQC; similarly, $\overline{CH}$ bisects $\angle SCB$. □



Next we have the following angle chasing claim.

Claim. We have $TPSQ$ cyclic.

Proof. Note that

$$\begin{aligned} \measuredangle PTQ &= 2\measuredangle PAQ = 2\measuredangle BAC \\ \measuredangle PSQ &= \measuredangle CSB = -(\measuredangle SBC + \measuredangle BCS) = \measuredangle CBQ + \measuredangle PCB \\ &= -2\measuredangle QCB - 2\measuredangle CBP = -2\measuredangle ACB - 2\measuredangle CBA = 2\measuredangle BAC. \end{aligned}$$

Hence $\measuredangle PTQ = \measuredangle PSQ$ as needed. □

Hence from $TP = TQ$ we see $\overline{ST}$ is a bisector of $\angle QSP$. Since $\angle QSP > 90^\circ$, it follows $\overline{ST}$ is an interior angle bisector of $\angle QSP$. This concludes the proof.