EGMO 2014/4

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Twitch Solves ISL

Episode 147

Problem

Determine all positive integers $n \geq 2$ for which there exist integers $x_1, x_2, \ldots, x_{n-1}$ satisfying the condition that if $0 < i < n$, $0 < j < n$, $i \neq j$ and n divides $2i + j$, then $x_i < x_j$.

Video

https://youtu.be/Ce0D5Uzxnew

External Link

https://aops.com/community/p3460731

Solution

The answer is $n = 2^k$ and $n = 3 \cdot 2^k$, for each $k \geq 0$ (excluding $n = 1$).

We work with the set $S = \{1, 2, \dots, n-1\}$ mod n of nonzero residues modulo n instead. We define the relation $\prec$ on S to mean that $2i + j \equiv 0 \pmod n$ and $i \neq j$, for $i, j \in S$. Then the problem satisfies the conditions if and only if $\prec$ has no cycles, i.e. $\prec$ imposes a partial order on S.

The existence of a cycle for $\prec$ is equivalent to some choice of $t_1 \in S$ and an integer $m \geq 2$ such that

$$t_1 \prec t_2 \prec \dots \prec t_m \prec t_1.$$

Unwinding the definition, this is equivalent to two conditions:

- We need $t_i \not\equiv t_{i+1} \pmod n$ for $i = 1, \dots, m$ (where $t_{m+1} = t_1$). This is equivalent to
$$3 \cdot 2^{i-1} \cdot t_1 \equiv 0 \pmod n \qquad (\heartsuit).$$
- For $t_m \prec t_1$ to be true, we need
$$(-2)^m t_1 \equiv t_1 \pmod n \iff ((-2)^m - 1)\, t_1 \equiv 0 \pmod n. \qquad (\spadesuit)$$

We now analyze three cases:

- Let $n = 2^k$. Suppose for contradiction some cycle exists. Then $(-2)^m - 1$ is coprime to n, so $(\spadesuit)$ would imply $t_1 \equiv 0 \pmod n$, contradiction.
- Let $n = 3 \cdot 2^k$. Suppose for contradiction some cycle exists. If $(\spadesuit)$ holds for some m, then $2^k \mid t_1$, so the only possibility is that $t_1 \equiv \pm 2^k \pmod n$ and $3 \mid (-2)^m - 1$. However, in that case $(\heartsuit)$ is violated for $i = 1$, contradiction.
- Suppose n had an n has an odd divisor $d \mid n$ and $d \geq 5$. Then taking $t_1 = n/d$ and $m = \varphi(d)$, the equation $(\spadesuit)$ is true. Moreover, $(\heartsuit)$ is true because there is at least one odd prime p with $\nu_p(n) > \nu_p(3t_1) = \nu_p(3n/d)$ (since $d \geq 5$ is odd). So indeed it's possible to construct a cycle.

Thus these are all the answers and the only answers.