EGMO 2012/3

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Twitch Solves ISL

Episode 147

Problem

Solve over $\mathbb{R}$ the functional equation

$$f\left(yf(x+y)+f(x)\right)=4x+2yf(x+y).$$

Video

https://youtu.be/a7xufmOsa9U

External Link

https://aops.com/community/p2658967

Solution

The only solution is $f(x) \equiv 2x$ which obviously works.

Let $P(x, y)$ be the given condition. Then:

- Note $P(x, 0) \implies f(f(x)) = 4x$; in particular f is bijective.
  - ...This also implies $f(4x) = f(f(f(x))) = 4f(x)$.
  - Taking $x = 0$ gives $f(0) = 4f(0) \implies f(0) = 0$.
- Now $P(0, 2) \implies f(2f(2)) = 4f(2) = f(8) \implies f(2) = 4$.
- Then $P(0, 1) \implies f(f(1)) = 2f(1) \implies 4 = 2f(1) \implies f(1) = 2$.
- Finally, $P(x, 1 - x)$ gives

$$f\left(2(1 - x) + f(x)\right) = 4x + 4(1 - x) = 4.$$

  Since f is a bijection and $f(2) = 4$, this means $2(1 - x) + f(x) = 2$, so $f(x) = 2x$ as desired.