# EGMO 2024/5

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Episode 146

## Problem

Find all functions $f\colon \mathbb{N} \to \mathbb{N}$ such that for all $x, y \in \mathbb{N}$:

(i) $x$ and $f(x)$ have the same number of positive divisors.

(ii) if $x \nmid y$ and $y \nmid x$, then:

$$\gcd(f(x), f(y)) > f(\gcd(x, y)).$$

## Video

https://youtu.be/Yh0r9zHgXYA

## External Link

https://aops.com/community/p30440446

## Solution

The answer is
$$f(x) = \ell^{d(x)-1}$$
where $\ell$ is a fixed prime, and $d(\bullet)$ is the divisor counting function. This works, because when $x \nmid y$ and $y \nmid x$ we obviously have $d(\gcd(x,y)) < \min(d(x), d(y))$. So now we prove this is the only solution.

Obviously $f(1) = 1$; we ignore this case further.

Note that $f(p)$ is always a prime for each prime $p$. But if $p$ and $q$ are two different primes, then apparently
$$\gcd\left(f(p), f(q)\right) > f(\gcd(p,q)) = f(1) \geq 1$$
and yet the left-hand side is the GCD of two primes. So this could only happen if they are the *same* prime; hence we conclude $f(p)$ is constant, say $\ell$.

**Claim.** Every output of $f$ besides $f(1)$ must be a multiple of $\ell$

*Proof.* To show $\ell \mid f(x)$ for $x > 1$, take $y$ to be any prime not dividing $x$. Then $\gcd(f(x), \ell) > 1$, as needed. □

This allows us to extend our earlier claim as follows:

**Claim.** If $p_1, \dots, p_k$ are distinct primes and $q_1, \dots, q_k$ are primes then
$$f\left(p_1^{q_1-1} \dots p_k^{q_k-1}\right) = \ell^{q_1 \cdots q_k - 1}.$$

*Proof.* By induction on $k$. For the base case $k = 1$, we know $f(p^{q-1})$ needs to be a multiple of $\ell$, and the only multiple of $\ell$ with $q$ prime divisors is $\ell^{q-1}$.

Suppose $k \geq 2$ and WLOG $q_1 \geq q_2 \geq \dots \geq q_k$. We let $p$ be any other prime. Then
$$\gcd\left(f(p_1^{q_1-1} \dots p_k^{q_k-1}), f(p_1^{q_1-1} \dots p_{k-1}^{q_{k-1}-1} p)\right) > f\left(p_1^{q_1-1} \dots p_{k-1}^{q_{k-1}-1}\right) = q_1 q_2 \dots q_{k-1}.$$
Hence $f(p_1^{q_1-1} \dots p_k^{q_k-1})$ is divisible by $\ell^{q_1 \cdots q_{k-1}}$. There are no proper divisors of $q_1 \dots q_k$ exceeding $q_1 \dots q_{k-1}$, so this forces $f(p_1^{q_1-1} \dots p_k^{q_k-1}) = \ell^{q_1 \cdots q_k - 1}$ exactly. □

We will now attack the main problem in the following presentation:

**Claim.** Suppose $p_1, \dots, p_k, r_1, \dots, r_m$ are pairwise distinct primes, and $q_1, \dots, q_k$ are also prime numbers, and $e_1, \dots, e_m \geq 2$. Let
$$x = p_1^{q_1-1} \dots p_k^{q_k-1} r_1^{e_1-1} \dots r_m^{e_m-1}$$
and set
$$n = d(x) = q_1 \dots q_k e_1 \dots e_m.$$
Then $f(x) = \ell^{n-1}$.

*Proof.* The proof is by induction on $m$, with the base case $m = 0$ already done. For the inductive step, look at $e_m$. If $e_m$ is prime there is nothing to do. Otherwise, $e_m \geq 3$, and we can take a prime $q$ satisfying
$$\frac{e_m}{2} < q < e_m.$$

and let $z$ be a random other prime not appearing already; then choosing

$$\begin{aligned} y &= p_1^{q_1-1} \dots p_k^{q_k-1} r_1^{e_1-1} \dots r_{m-1}^{q-1} z \\ \gcd(x,y) &= p_1^{q_1-1} \dots p_k^{q_k-1} r_1^{e_1-1} \dots r_{m-1}^{q-1}. \end{aligned}$$

Then

$$\begin{aligned} f(y) &= \ell^{2q_1 \dots q_k \cdot e_1 \dots e_{m-1} \cdot q - 1} > \ell^{n-1} \\ f(\gcd(x,y)) &= \ell^{q_1 \dots q_k \cdot e_1 \dots e_{m-1} \cdot q - 1} = \ell^{n \cdot \frac{q}{e_m} - 1} > \ell^{\frac{n}{2}-1}. \end{aligned}$$

It follows that $\nu_\ell(f(x)) + 1 > \frac{n}{2}$. We know $n = d(f(x)) = \prod_p (\nu_p(f(x)) + 1)$. If a number greater than $n/2$ appears on the product of the RHS, then in fact the product consists only of $n$. This completes the induction. □