EGMO 2024/4

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Twitch Solves ISL

Episode 146

Problem

For a sequence $a_1 < a_2 < \cdots < a_n$ of integers, a pair (a_i, a_j) with $1 \le i < j \le n$ is called *interesting* if there exists a pair (a_k, a_ℓ) of integers with $1 \le k < \ell \le n$ such that

$$\frac{a_\ell - a_k}{a_j - a_i} = 2.$$

For each $n \ge 3$, find the largest possible number of interesting pairs in a sequence of length n.

Video

https://youtu.be/AawAbKQsCVM

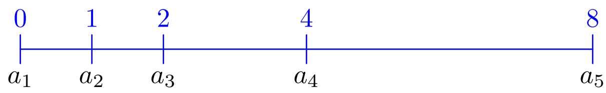
External Link

https://aops.com/community/p30440509

Solution

The answer is $\binom{n}{2} - (n-2)$.

Construction. We take $a_1 = 0$ and $a_i = 2^{i-2}$ for $i \geq 2$. For example, when $n = 5$ we get the following figure:



We see that all the pairs (i, j) are interesting *except* for the cases where $j = n$ and $i \in \{1, 2, \dots, n-2\}$. Indeed:

- If $i = 1$ and $j < n$, then $a_j - a_i = 2^{j-2} = \frac{a_{j+1} - a_j}{2}$.
- If $i > 1$ and $j < n$, then $a_j - a_i = \frac{2a_j - 2a_i}{2} = \frac{a_{j+1} - a_{i+1}}{2}$.
- If $i = n-1$ and $j = n$, then $a_j - a_i = 2^{n-3} = \frac{a_n - a_1}{2}$.

Proof of bound. Let $m = \frac{a_1 + a_n}{2}$. Notice that

- If $a_j > m$, then $(1, j)$ is never interesting.
- If $a_i < m$, then (i, n) is never interesting.

This eliminates at least $(n-1) - 1$ pairs, one for each index k with $a_k \neq m$. (The pair $(1, n)$ is counted twice here.) This gives the claimed bound.