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Twitch Solves ISL

Episode 146

Problem

We call a positive integer n *peculiar* if, for any positive divisor d of n the integer $d(d+1)$ divides $n(n+1)$. Prove that for any four different peculiar positive integers A, B, C and D, the following holds:

$$\gcd(A, B, C, D) = 1.$$

Video

https://youtu.be/EJS9br0gyqE

External Link

https://aops.com/community/p30433680

Solution

Note that 1 and any prime are peculiar. We classify all composite peculiar numbers in a series of claims.

Claim. A peculiar number n has at most two prime factors.

Proof. Let p be the smallest prime dividing n, and let $\frac{n}{p} = c$. Then

$$\frac{n}{p} \cdot \left(\frac{n}{p} + 1\right) \mid n(n+1)$$

In particular,

$$c + 1 \mid cp \cdot (cp + 1).$$

However,

$$cp \cdot (cp + 1) \equiv p(p - 1) \pmod{c + 1}.$$

So, since $p(p - 1) \neq 0$, we get a bound

$$c \leq p^2 - p \implies n = cp \leq p^3 - p^2 < p^3.$$

Hence, $n < p^3$ and p is the smallest prime dividing n, n can have at most one additional prime factor. □

Claim. The square of a prime is never peculiar.

Proof. If $n = p^2$ we need $p(p + 1) \mid p^2(p^2 + 1)$, or $p + 1 \mid p^2 + 1$, which never holds as $p^2 + 1 \equiv 2 \pmod{p + 1}$. □

Claim. If $n = pq$ is peculiar for primes $p > q$, then $p = (q + 1)(q - 2) + 1$.

Proof. Note that 6 is not peculiar (as $3 \cdot 4 \nmid 6 \cdot 7$). Assume $n > 6$ and write the equations

$$\begin{aligned} p(p+1) \mid pq(pq+1) &\iff p+1 \mid q(pq+1) \iff p+1 \mid q(q-1) \\ q(q+1) \mid pq(pq+1) &\iff q+1 \mid p(pq+1) \iff q+1 \mid p(p-1) \end{aligned}$$

In the second equation, since $p > q + 1$, we find $\gcd(q + 1, p) = 0$ so

$$p \equiv 1 \pmod{q + 1}.$$

So let $p = 1 + k(q + 1)$. On the other hand, we also note that

$$2 + k(q + 1) = p + 1 \leq q(q - 1)$$

and hence $k < q - 1$; that is, $k \in \{1, 2, \ldots, q - 2\}$.

Now, if $p + 1 = 2 + k(q + 1)$ is divisible by q, it follows $k \equiv -2 \pmod{q}$ and therefore $k = q - 2$ exactly. If it isn't, then we would have $p + 1 \mid q - 1$ which is impossible. So the claim is proved. □

Hence, given a fixed prime ℓ, there are at most three peculiar numbers divisible by ℓ: namely

- ℓ itself;
- $\ell \cdot [(\ell + 1)(\ell - 2) + 1]$, if the bracketed number is indeed prime;
- $\ell \cdot r$, if there is indeed a prime r such that $\ell = (r + 1)(r - 2) + 1$.

Hence given four distinct peculiar numbers, they can have no common prime factor.