EGMO 2024/1

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Twitch Solves ISL

Episode 146

Problem

Two different integers u and v are written on a board. We perform a sequence of steps. At each step we do one of the following two operations:

(i) If a and b are different integers on the board, then we can write $a + b$ on the board, if it is not already there.

(ii) If a, b and c are three different integers on the board, and if an integer x satisfies $ax^2 + bx + c = 0$, then we can write x on the board, if it is not already there.

Determine all pairs of starting numbers (u, v) from which any integer can eventually be written on the board after a finite sequence of steps.

Video

https://youtu.be/wb-9khsmdpA

External Link

https://aops.com/community/p30433696

Solution

Note that the following cases are impossible:

- If either $u = 0$ or $v = 0$, operation (i) cannot produce numbers, and (ii) cannot be used at all. Hence the task is obviously impossible.
- If $\{u, v\} = \{-1, 1\}$, operation (i) produces 0, and no more numbers can be added with operation (i) or (ii). So this case cannot be done.
- If $u, v < 0$, then we claim all numbers written are negative. Indeed, (i) respects this; and also, if $a, b, c < 0$ then no $x > 0$ can obey $ax^2 + bx + c = 0$. So the task is impossible here too.

We claim the task is possible in all other cases.

Claim. We can write -1.

Proof. We can write $u+v$, and the three numbers u, v, $u+v$ are different. The quadratic $ux^2 + (u + v)x + v$ has root -1. □

Claim. We can write a positive number $m \geq 2$ on the board.

Proof. We assumed that at least one of u and v; say $u = \max(u, v) > 0$. If $u \geq 2$ we're done. Suppose $u = 1$; then v is some negative number with $v \leq -2$. We can write $(-1) + 1 = 0$. Then $0x^2 + x + v = 0$ has root $-v$, which is the desired number. □

Let $m \geq 2$ be from the claim; then we can write $m - 1 \geq 1$, and hence we can write all of $m - 1$, $2m - 1$, $3m - 1$, $\dots$. Thus (because we have -1) we can then write any nonnegative integer by adding -1 several times. (However, we cannot write -2 in this way, because we cannot add -1 to itself.)

But now for $n > 0$, $-n$ is a root of $0x^2 + x + n$, so we can write the negative integers too.