Twitch 145.1

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Twitch Solves ISL

Episode 145

Problem

Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that for all $x, y \in \mathbb{R}$, we have:

$$f(x) = f(x + y + f(y)).$$

Video

https://youtu.be/tc9C60_2-KM

Solution

Let S be any $\mathbb{Z}$-submodule (additive subgroup) of $\mathbb{R}$. For each coset $T \in \mathbb{R}/S$ we choose a representative $\rho(T)$ of it; in other words we pick an arbitrary choice function

$$f(x) = -\rho(x)$$

is a solution to the given equation, and moreover all solutions are of this form.

- To check these work, note that $y+f(y) = y-\rho(y) \in S$, hence $\rho(x) = \rho(x+y-\rho(y))$, so they work.
- Conversely, given any function f satisfying the given equation, if we let

  $$S \coloneqq \langle y + f(y) \mid y \in \mathbb{R} \rangle_{\mathbb{Z}}$$

  then f is invariant under shifting by S, i.e. $f(x+s) = f(x)$ for all $x \in \mathbb{R}$ and $s \in S$. In other words, $f(x)$ only depends on $x \pmod S$. Since $x + f(x) \in S$ by construction, this finishes the proof.