Taiwan MO 2024/4

Evan Chen

Twitch Solves ISL

Episode 145

Problem

Suppose O is the circumcenter of $\triangle ABC$, and E and F are points on segments CA and AB respectively with $E, F \neq A$. Let P be a point such that $PB = PF$ and $PC = PE$. Let OP intersect CA and AB at points Q and R respectively. Let the line passing through P and perpendicular to EF intersect CA and AB at points S and T respectively. Prove that points Q, R, S, and T are concyclic.

Video

https://youtu.be/eiHul2kJINE

External Link

https://aops.com/community/p29778110

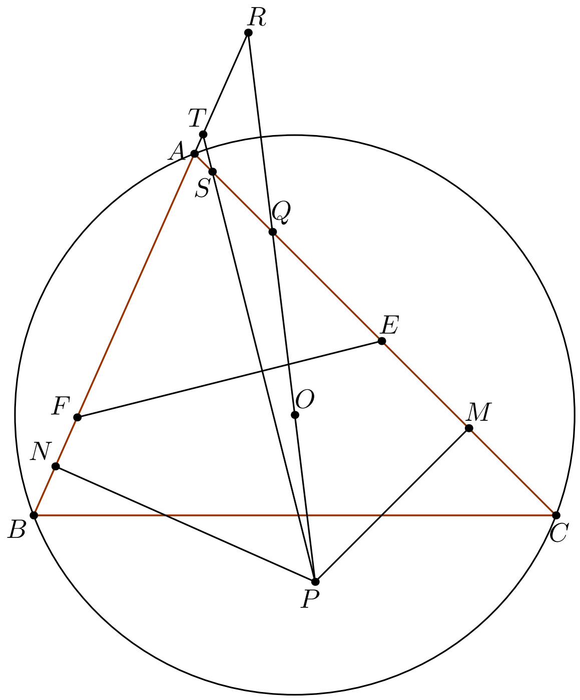
Solution

Note that

$$\measuredangle QST = \measuredangle(\overline{PST}, \overline{AC}) = 90^\circ + \measuredangle(\overline{EF}, \overline{AC})$$
$$\measuredangle QRT = \measuredangle(\overline{AB}, \overline{OP}).$$

So we can erase all the points Q, R, S, T from the picture and focus just on proving the claim that:

Claim. $\measuredangle(\overline{AB}, \overline{OP}) = \measuredangle(\overline{EF}, \overline{AC}) + 90^\circ$.



Proof. Use complex numbers with ABC the unit circle. Let P be any point, with coordinate p. Then the foot from P to $\overline{AC}$ has coordinates

$$M \coloneqq \frac{1}{2}\left(p + a + c - ac\overline{p}\right)$$

and so (since M is the midpoint of $\overline{EC}$) we get

$$E = 2 \cdot M - c = p + a - ac\overline{p}.$$

Analogously

$$F = p + a - ab\overline{p}.$$

So then we have the ratio

$$z \coloneqq \frac{E-F}{a-c} \div \frac{a-b}{p-0} = \frac{a\overline{p}(b-c) \cdot p}{(a-b)(a-c)} = p\overline{p} \cdot \frac{a(b-c)}{(a-b)(a-c)}.$$

The complex conjugate is

$$\bar{p}p \cdot \frac{\frac{1}{a}\left(\frac{1}{b} - \frac{1}{c}\right)}{\left(\frac{1}{a} - \frac{1}{b}\right)\left(\frac{1}{a} - \frac{1}{c}\right)} = -z.$$

so z is pure imaginary and we're done. □