Frog jump

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Twitch Solves ISL

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Problem

There is a hidden infinite binary string $a_1 a_2 a_3 \dots \in \{0,1\}^\infty$ such that $\{t \in \mathbb{N} : a_t = 1\}$ has upper density zero. The digits a_t are revealed in order. For each $t \geq 1$, before a_t is revealed, a frog decides to wait or cross the road. Then a_t is revealed. If the frog crossed the road and $a_t = 1$, then the frog is squashed. If the frog crossed the road and $a_t = 0$ then the frog has safely crossed the road. Otherwise the process repeats with a_{t+1}.

Determine whether one can find a strategy such that, regardless of which string $a_1 a_2 a_3 \dots$ is chosen, the frog successfully crosses the road with probability at least 99%.

Video

https://youtu.be/W2chl6FnsPg

External Link

https://aops.com/community/p27344416

Solution

Answer: yes.

Fix absolute constants $\delta, \varepsilon > 0$.

Let's say an index $\ell \in \mathbb{Z}_{\geq 0}$ is *bad* if the density of 1's in $a_{2^\ell} \dots a_{2^{\ell+1}-1}$ is more than δ; we also say the interval $[2^\ell, 2^{\ell+1} - 1]$ is bad.

Claim. There are only finitely many bad indices.

Proof. If ℓ is a bad index, then the density of 1's in $\{a_1, \dots, a_{2^{\ell+1}-1}\}$ is at least $\delta/2$. $\square$

The frog then employs the following strategy:

- Initialize $p = \varepsilon$, $\ell = 0$, $t = 2^0$, just before a_1 is revealed.
- With probability p, commit to jumping at a uniform random index in the set $\{2^\ell, \dots, 2^{\ell+1} - 1\}$ (ignoring any future data).
- Otherwise, commit to not jumping for $t = 2^\ell, \dots, 2^{\ell+1} - 1$. In that case, if the interval $[2^\ell, 2^{\ell+1} - 1]$ turned out to be bad, halve the value of p.

Let $L = \{\ell_0, \dots, \ell_r\}$ be the finite set of bad indices. The probability that the frog jumps in the bad interval $[2^{\ell_i}, 2^{\ell_i+1} - 1]$ is at most $\frac{\varepsilon_0}{2^i}$ by the way the algorithm is set. Consequently, the total probability of a jump inside any bad interval is at most

$$\sum_{i \geq 0} \frac{\varepsilon}{2^i} = 2\varepsilon.$$

On the other hand, the probability of death when jumping inside a good interval is also at most δ. And the frog jumps at some point with probability 1.

Hence, the probability of success is at least $1 - (2\varepsilon + \delta)$. Therefore, picking e.g. $\varepsilon = \delta = \frac{1}{300}$ solves the problem.