XOOK 2023/2

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Twitch Solves ISL

Episode 144

Problem

Let ABC be a triangle with incenter I, which has foot D to BC, and let T be the A-mixtillinear touch point. Let M be the midpoint o f BC, which has foot X to line AI. Let AT intersect (DXT) again at Q. Show that (AXQ) is tangent to the incircle.

Video

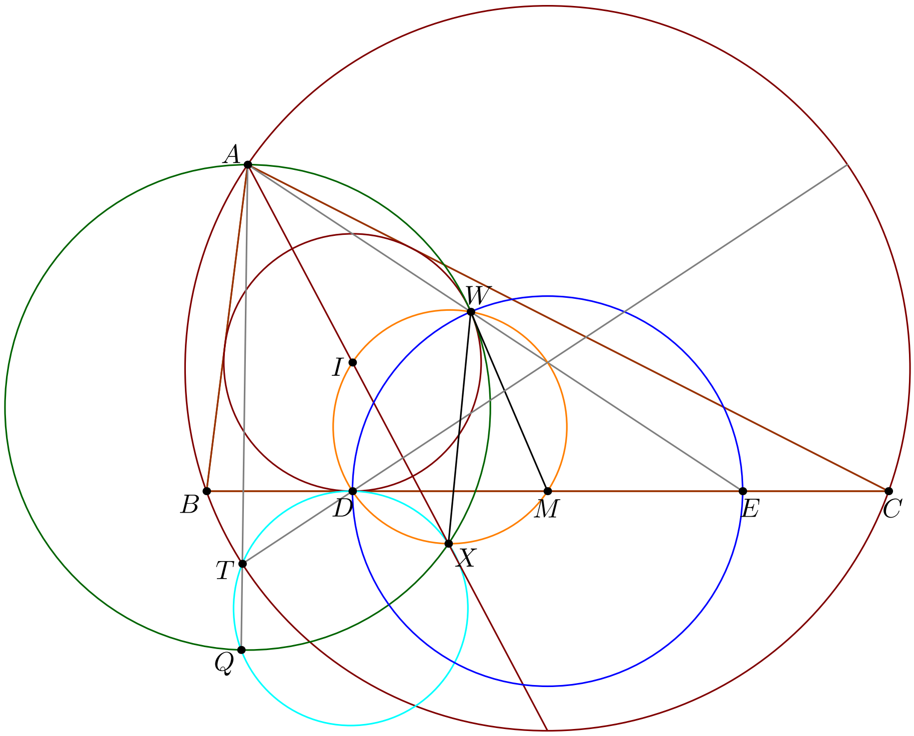
https://youtu.be/cuUQkgZ4xVc

External Link

https://aops.com/community/p29571593

Solution

Let E be the contact point of the A-excircle, so $\overline{ATQ}$ and $\overline{AE}$ are isogonal. The circle centered at M with radius $MD = ME$ intersects the incircle again at a point W such that $\angle DWE = 90^\circ$; this implies A, W, E are collinear, $MW = MD = ME$, and $\overline{MW}$ is tangent to the incircle. (See USA TST 2015/1 for a figure with the same circle centered at M.)



Claim. $WIDXM$ is cyclic with diameter $\overline{IM}$.

Proof. Because $\measuredangle IDM = \measuredangle IXM = \measuredangle IWM = 90^\circ$. □

Claim. $\overline{MW}$ is also tangent to (AWX).

Proof. $\measuredangle WXA = \measuredangle WXI = \measuredangle WDI = \measuredangle WED = \measuredangle MWE = \measuredangle MWA$. □

Claim. Q lies on (AXW).

Proof. Note that

$$\begin{aligned} \measuredangle XQT = \measuredangle XDT &= \measuredangle XDM + \measuredangle(\overline{MD}, \overline{DT}) = \measuredangle XWM + \measuredangle(\overline{AE}, \overline{EM}) \\ &= \measuredangle XWM + \measuredangle WEM = \measuredangle XWM + \measuredangle MWE \\ &= \measuredangle XWE = \measuredangle XWA. \end{aligned}$$

□