H2716390

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Twitch Solves ISL

Episode 144

Problem

Let a and b be fixed positive integers. We say that a prime p is fun if there exists a positive integer n satisfying the following conditions:

- p divides $a^{n!} + b$.
- p divides $a^{(n+1)!} + b$.
- $p < 2n^2 + 1$.

Show that there are finitely many fun primes.

Video

https://youtu.be/G_cRUZ1TEmU

External Link

https://aops.com/community/p23622966

Solution

We will consider $n > 2^{100}$, since this adds at most finitely many primes. We also assume $p \nmid ab$ throughout, as well as $p > b + 1$.

Note that we have

$$a^{n!} \equiv -b \pmod{p} \implies a^{n!} \not\equiv 1 \pmod{p}$$

because we assume $p > b + 1$. However, we also get

$$a^{n \cdot n!} \equiv 1 \pmod{p}.$$

Thus, if e denotes the order of $a \pmod{p}$, then $e \nmid n!$, but $e \mid n \cdot n!$. So there exists a prime q with $q \mid n$ such that

$$\nu_q(n!) < \nu_q(e) \leq \nu_q(n!) + \nu_q(n).$$

We also know that

$$e \mid p - 1 < 2n^2.$$

Claim. We have $n = q$.

Proof. Assume not, meaning $q \leq n/2$. Start by using the estimate

$$n^{2.1} > 2n^2 > q^{\nu_q(e)} \geq q^{\nu_q(n!)+1} \geq q^{\frac{n}{q}+1}.$$

In particular, we certainly need $n^{2.1} \geq q^3$, so $q < n^{0.7}$. Using that, we can further estimate

$$n^{2.1} \geq 2^{\frac{n}{q}+1} \geq 2^{n^{0.3}+1}$$

which is false for $n > 2^{100}$. $\square$

So e must be a multiple of n^2, but $e < 2n^2$, so in fact $e = n^2$ exactly. That means $p = n^2 + 1$. However p and n are both primes, so this is a contradiction.

Remark. The same proof shows there are finitely many fun primes if the condition $a, b > 0$ is relaxed to a and b are nonzero integers with $b \neq -1$.