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Twitch Solves ISL

Episode 143

Problem

Let m and n be positive integers. A circular necklace contains mn beads, each either red or blue. It turned out that no matter how the necklace was cut into m blocks of n consecutive beads, each block had a distinct number of red beads. Determine, with proof, all possible values of the ordered pair (m, n).

Video

https://youtu.be/ZLOghPRDgoU

External Link

https://aops.com/community/p30227198

Solution

The answer is $m \le n+1$ only.

Proof the task requires $m \le n+1$. Each of the m blocks has a red bead count between 0 and n, each of which appears at most once, so $m \le n+1$ is needed.

Construction when $m = n+1$. For concreteness, here is the construction for $n = 4$, which obviously generalizes. The beads are listed in reading order as an array with $n+1$ rows and n columns. Four of the blue beads have been labeled $B_1, \dots, B_n$ to make them easier to track as they move.

$$T_0 = \begin{bmatrix} R & R & R & R \\ R & R & R & B_1 \\ R & R & B & B_2 \\ R & B & B & B_3 \\ B & B & B & B_4 \end{bmatrix}.$$

To prove this construction works, it suffices to consider the n cuts $T_0, T_1, T_2, \dots, T_{n-1}$ made where T_i differs from T_{i-1} by having the cuts one bead later also have the property each row has a distinct red count:

$$T_1 = \begin{bmatrix} R & R & R & R \\ R & R & B_1 & R \\ R & B & B_2 & R \\ B & B & B_3 & B \\ B & B & B_4 & R \end{bmatrix} \quad T_2 = \begin{bmatrix} R & R & R & R \\ R & B_1 & R & R \\ B & B_2 & R & B \\ B & B_3 & B & B \\ B & B_4 & R & R \end{bmatrix} \quad T_3 = \begin{bmatrix} R & R & R & R \\ B_1 & R & R & B \\ B_2 & R & B & B \\ B_3 & B & B & B \\ B_4 & R & R & R \end{bmatrix}.$$

We can construct a table showing for each $1 \le k \le n+1$ the number of red beads which are in the $(k+1)$st row of T_i from the bottom:

k	T_0	T_1	T_2	T_3
$k=4$	4	4	4	4
$k=3$	3	3	3	2
$k=2$	2	2	1	1
$k=1$	1	0	0	0
$k=0$	0	1	2	3

This suggests following explicit formula for the entry of the (i,k)th cell which can then be checked straightforwardly:

$$\#(\text{red cells in } k\text{th row of } T_i) = \begin{cases} k & k > i \\ k-1 & i \ge k > 0 \\ i & k = 0. \end{cases}$$

And one can see for each i, the counts are all distinct (they are $(i, 0, 1, \dots, k-1, k+1, \dots, k)$ from bottom to top). This completes the construction.

Construction when $m < n+1$. Fix m. Take the construction for $(m, m-1)$ and add $n+1-m$ cyan beads to the start of each row; for example, if $n = 7$ and $m = 5$ then the new construction is

$$T = \begin{bmatrix} C & C & C & R & R & R & R \\ C & C & C & R & R & R & B_1 \\ C & C & C & R & R & B & B_2 \\ C & C & C & R & B & B & B_3 \\ C & C & C & B & B & B & B_4 \end{bmatrix}.$$

This construction still works for the same reason (the cyan beads do nothing for the first $n+1-m$ shifts, then one reduces to the previous case). If we treat cyan as a shade of blue, this finishes the problem.