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Twitch Solves ISL

Episode 142

Problem

Solve over $\mathbb{R}$ the functional equation $f(x^2 - y) + 2yf(x) = f(f(x)) + f(y)$.

Video

https://youtu.be/yLhlJHJDxvQ

External Link

https://aops.com/community/p30227204

Solution

The answer is $f(x) \equiv x^2$, $f(x) \equiv 0$, $f(x) \equiv -x^2$, which obviously work.

Let $P(x,y)$ be the usual assertion.

Claim. We have $f(0) = 0$ and f even.

Proof. Combine $P(1, 1/2)$ with $P(1, 0)$ to get $f(0) = 0$. Use $P(0, y)$ to deduce f is even. $\square$

Claim. $f(x) \in \{-x^2, 0, x^2\}$ for every $x \in \mathbb{R}$.

Proof. Note that $P(x, x^2/2)$ and $P(x, 0)$ respectively give

$$x^2 f(x) = f(x^2) = f(f(x)).$$

Repeating this key identity several times gives

$$\begin{aligned} f(f(f(x))) &= f(f(x^2)) = f(x^4) = x^4 f(x^2) \\ &= f(x)^2 \cdot f(f(x)) = f(x)^2 f(x^2) = f(x)^3 x^2 \end{aligned}$$

Suppose $t \neq 0$ is such that $f(t^2) \neq 0$. Then the above equalities imply

$$t^4 f(t^2) = f(t)^2 f(t^2) \implies f(t) = \pm t^2$$

and then

$$f(t)^2 f(t^2) = f(t)^3 t^2 \implies f(t^2) = \pm t^2.$$

Together with f even, we get the desired result. $\square$

Remark. Another proof is possible here that doesn't use as iterations of f: the idea is to "show f is injective up to sign outside its kernel". Specifically, if $f(a) = f(b) \neq 0$, then $a^2 f(a) = f(f(a)) = f(f(b)) = b^2 f(b) \implies a^2 = b^2$. But we also have $f(f(x)) = f(x^2)$, so we are done except in the case $f(f(x)) = f(x^2) = 0$. That would imply $x^2 f(x) = 0$, so the claim follows.

Now, note that $P(1, y)$ gives

$$f(1-y) + 2y \cdot f(1) = f(1) + f(y).$$

We consider cases on $f(1)$ and show that f matches the desired form.

- If $f(1) = 1$, then $f(1-y) + (2y-1) = f(y)$. Consider the nine possibilities that arise:

  $$\begin{array}{lll} (1-y)^2 + (2y-1) = y^2 & 0 + (2y-1) = y^2 & -(1-y)^2 + (2y-1) = y^2 \\ (1-y)^2 + (2y-1) = 0 & 0 + (2y-1) = 0 & -(1-y)^2 + (2y-1) = 0 \\ (1-y)^2 + (2y-1) = -y^2 & 0 + (2y-1) = -y^2 & -(1-y)^2 + (2y-1) = -y^2. \end{array}$$

  Each of the last eight equations is a nontrivial polynomial equation. Hence, there is some constant $C > 100$ such that the latter eight equations are all false for any real number $y > C$. Consequently, $f(y) = y^2$ for $y > C$.

  Finally, for any real number $z > 0$, take $x, y > C$ such that $x^2 - y = z$; then $P(x, y)$ proves $f(z) = z^2$ too.

- Note that (as f is even), f works iff $-f$ works, so the case $f(1) = -1$ is analogous.

- If $f(1) = 0$, then $f(1-y) = f(y)$. Hence for any y such that $|1-y| \neq |y|$, we conclude $f(y) = 0$. Then take $P(2, 7/2) \implies f(1/2) = 0$.

Remark. There is another clever symmetry approach possible after the main claim. The idea is to write

$$P(x, y^2) \implies f(x^2 - y^2) + 2y^2 f(x) = f(f(x)) + f(f(y)).$$

Since f is even gives $f(x^2 - y^2) = f(y^2 - x^2)$, one can swap the roles of x and y to get $2y^2 f(x) = 2x^2 f(y)$. Set $y = 1$ to finish.