JMO 2024/4

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Twitch Solves ISL

Episode 142

Problem

Let $n \geq 3$ be an integer. Rowan and Colin play a game on an $n \times n$ grid of squares, where each square is colored either red or blue. Rowan is allowed to permute the rows of the grid and Colin is allowed to permute the columns. A grid coloring is orderly if:

- no matter how Rowan permutes the rows of the coloring, Colin can then permute the columns to restore the original grid coloring; and
- no matter how Colin permutes the columns of the coloring, Rowan can then permute the rows to restore the original grid coloring.

In terms of n, how many orderly colorings are there?

Video

https://youtu.be/s_bUukq7HiA

External Link

https://aops.com/community/p30227193

Solution

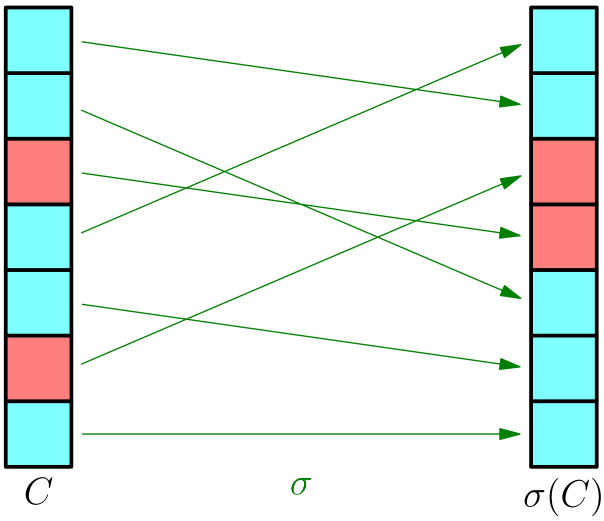
The answer is $2n! + 2$. In fact, we can describe all the orderly colorings as follows:

- The all-blue coloring.
- The all-red coloring.
- Each of the $n!$ colorings where every row/column has exactly one red cell.
- Each of the $n!$ colorings where every row/column has exactly one blue cell.

These obviously work; we turn our attention to proving these are the only ones.

For the other direction, fix a orderly coloring $\mathcal{A}$.

Consider any particular column C in $\mathcal{A}$ and let m denote the number of red cells that C has. Any row permutation (say σ) that Rowan chooses will transform C into some column $\sigma(C)$, and our assumption requires $\sigma(C)$ has to appear somewhere in the original assignment $\mathcal{A}$. An example for $n = 7$, $m = 2$, and a random σ is shown below.



On the other hand, the number of possible patterns of $\sigma(C)$ is easily seen to be exactly $\binom{n}{m}$ — and they must all appear. In particular, if $m \notin \{0, 1, n-1, n\}$, then we immediately get a contradiction because $\mathcal{A}$ would need too many columns (there are only n columns in $\mathcal{A}$, which is fewer than $\binom{n}{m}$). Moreover, if either $m = 1$ or $m = n-1$, then these columns are all the columns of $\mathcal{A}$; this leads to the $2n!$ main cases we found before.

The only remaining case is when $m \in \{0, n\}$ for every column, i.e. every column is monochromatic. It's easy to see in that case the columns must all be the same color.