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Episode 142

Problem

Find all positive integers k for which there is an irrational $\alpha > 1$ and a positive integer N such that $\lfloor \alpha^n \rfloor + k$ is a perfect square for every integer $n > N$.

Video

https://youtu.be/7wmIb_Byghc

External Link

https://aops.com/community/p24349942

Solution

The answer is $k = 3$ only.

Construction. Consider the integer sequence

$$x_n := \left(\frac{3+\sqrt{5}}{2}\right)^n + \left(\frac{3-\sqrt{5}}{2}\right)^n$$

defined by $x_0 = 2$, $x_1 = 3$, $x_2 = 7$, $x_{n+2} = 3x_{n+1} - x_n$, and so on. Then we have

$$x_n^2 := \left(\frac{14+6\sqrt{5}}{4}\right)^n + \left(\frac{14-6\sqrt{5}}{4}\right)^n + 2.$$

so

$$\left\lfloor \left(\frac{14+6\sqrt{5}}{4}\right)^n \right\rfloor = x_n^2 - 3$$

holds for every n (since $0 < \frac{14-6\sqrt{5}}{4} < 1$). This provides $\alpha = \frac{14+6\sqrt{5}}{4}$ as an example.

Proof that $k = 3$ is the only one possible. Throughout this solution, the big O notation treats α and k as $O(1)$.

For each $n > N$, we introduce the notation

$$\alpha^n = x_n^2 - k + \varepsilon_n \qquad x_n \in \mathbb{N}, 0 < \varepsilon_n < 1.$$

In particular, $x_n = \sqrt{\alpha^n + O(1)} = \alpha^{n/2} + O(\alpha^{-n/2})$.

We make the following assertion.

Claim. We have $1 - \varepsilon_n = O(\alpha^{-n})$ and $x_n^2 = x_{2n} + (k-1)$.

Proof. Note that

$$\begin{aligned} \alpha^{2n} &= x_{2n}^2 - k + \varepsilon_{2n} \\ \implies (\alpha^n - x_{2n})(\alpha^n + x_{2n}) &= -k + \varepsilon_{2n} \\ \implies \left(x_n^2 - x_{2n} - k + \varepsilon_n\right)(\alpha + x_{2n}) &= \varepsilon_{2n} - k \end{aligned}$$

Rearrange this to

$$x_n^2 - x_{2n} - k = -\varepsilon_n - \frac{k - \varepsilon_{2n}}{\alpha^n + x_{2n}}.$$

The right-hand side is an integer strictly between -2 and 0, so it's -1. So we get both the equality of the main terms

$$x_n^2 = x_{2n} + (k-1)$$

and the estimate on the error term

$$1 - \varepsilon_n = \frac{k - \varepsilon_{2n}}{\alpha^n + x_{2n}} = \frac{O(1)}{2\alpha^n + O(\alpha^{-n})} = O(\alpha^{-n}). \qquad \square$$

In the main equation, replacing x_n^2 with $x_{2n} + (k-1)$ gives

$$\alpha^n = x_{2n} - (1-\varepsilon_n) = x_{2n} - \frac{k-\varepsilon_{2n}}{\alpha^n + x_{2n}}.$$

The fraction equals $\frac{k-1+O(\alpha^{-n})}{2\alpha^n+O(\alpha^{-n})}$ and so we get

$$x_{2n} = \alpha^n + \frac{\frac{k-1}{2}}{\alpha^n} + O(\alpha^{-2n}).$$

Multiplying both sides by $\alpha + \frac{1}{\alpha}$ gives

$$\begin{aligned}\left(\alpha + \frac{1}{\alpha}\right) x_{2n} &= \alpha^{n+1} + \frac{\frac{k-1}{2}}{\alpha^{n+1}} + \alpha^{n-1} + \frac{\frac{k-1}{2}}{\alpha^{n-1}} + O(\alpha^{-2n}) \\ &= x_{2n+2} + x_{2n-2} + O(\alpha^{-2n}).\end{aligned}$$

Hence

$$\frac{x_{2n+2}+x_{2n-2}}{x_{2n}} = \alpha + \frac{1}{\alpha} + O(\alpha^{-3n}).$$

The critical claim is that the error term is actually zero exactly:

Claim. The equation

$$\frac{x_{2n+2}+x_{2n-2}}{x_{2n}} = \alpha + \frac{1}{\alpha}$$

holds for large enough n.

Proof. Notice that $\frac{x_{2n+2}+x_{2n-2}}{x_{2n}} - \frac{x_{2n}+x_{2n-4}}{x_{2n-2}}$ is the difference of two terms which are $O(\alpha^{-3n})$. However, it's the difference of two rational numbers whose denominators are each $\Theta(\alpha^n)$, ergo their difference is at least $1/O(\alpha^{2n})$. This is too large, so the error terms between consecutive fractions must be constant. Since it also decays to zero, it must be eventually zero. □

From this, we have a bona fide linear recurrence with exact equalities

$$x_{2n} = \lambda_+ \cdot \alpha^n + \lambda_- \cdot \alpha^{-n}$$

for some constants λ_+ and λ_-; but actually $\lambda_+ = 1$ and $\lambda_- = \frac{k-1}{2}$ to match our earlier equation for x_{2n}; that is, we have exactly

$$x_{2n} = \alpha^n + \frac{\frac{k-1}{2}}{\alpha^n}.$$

On the other hand, we are supposed to have $x_{2n}^2 = x_{4n} + (k-1)$ and comparing these two forces $\frac{k-1}{2} = 1$, so $k = 3$.