# USAMO 2024/5

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Twitch Solves ISL

Episode 141

## Problem

Point $D$ is selected inside acute triangle $ABC$ so that $\angle DAC = \angle ACB$ and $\angle BDC = 90^\circ + \angle BAC$. Point $E$ is chosen on ray $BD$ so that $AE = EC$. Let $M$ be the midpoint of $BC$. Show that line $AB$ is tangent to the circumcircle of triangle $BEM$.

## Video

https://youtu.be/7M-b5zXXpaY

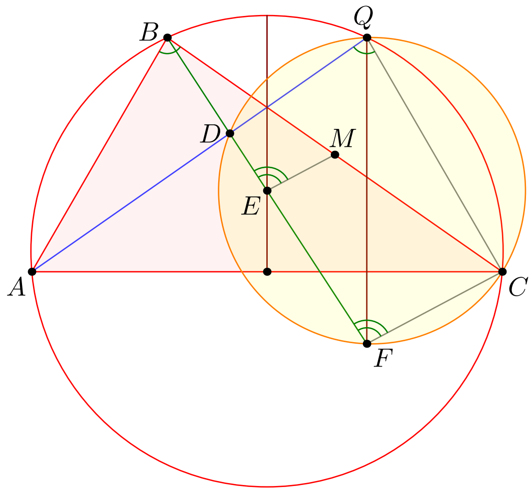
## External Link

https://aops.com/community/p30227196

## Solution

This problem has several approaches and we showcase a collection of them.

**The author's original solution.** Complete isosceles trapezoid $ABQC$ (so $D \in \overline{AQ}$). Reflect $B$ across $E$ to point $F$.




**Claim.** We have $DQCF$ is cyclic.

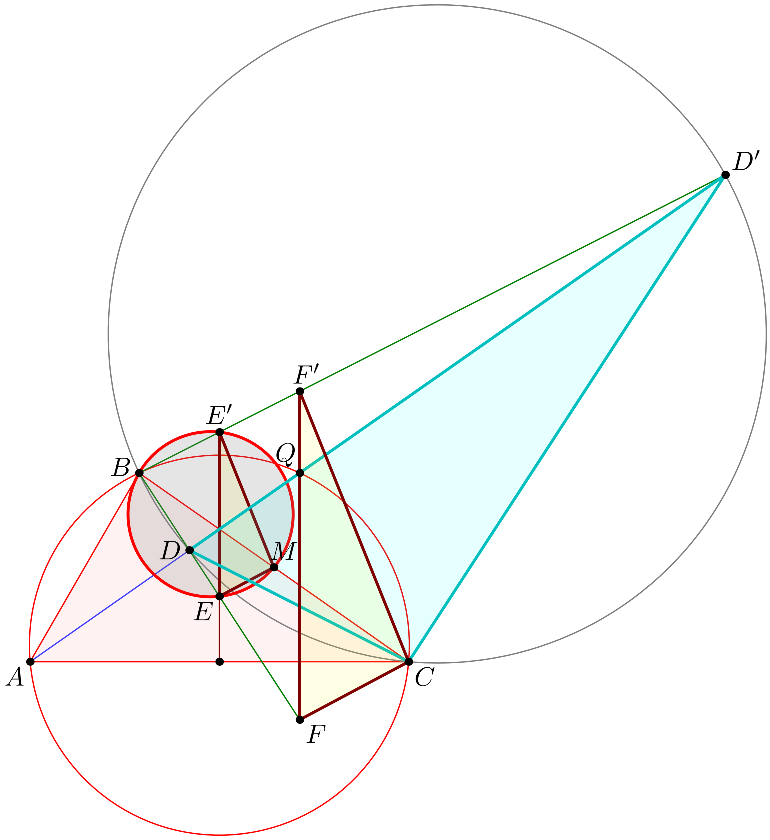
*Proof.* Since $EA = EC$, we have $\overline{QF} \perp \overline{AC}$ as line $QF$ is the image of the perpendicular bisector of $\overline{AC}$ under a homothety from $B$ with scale factor 2. Then

$$\begin{aligned} \measuredangle FDC &= -\measuredangle CDB = 180^\circ - (90^\circ + \measuredangle CAB) = 90^\circ - \measuredangle CAB \\ &= 90^\circ - \measuredangle QCA = \measuredangle FQC. \end{aligned} \qquad \square$$

To conclude, note that

$$\measuredangle BEM = \measuredangle BFC = \measuredangle DFC = \measuredangle DQC = \measuredangle AQC = \measuredangle ABC = \measuredangle ABM.$$

**Remark** (Motivation)**.** Here is one possible way to come up with the construction of point $F$ (at least this is what led Evan to find it). If one directs all the angles in the obvious way, there are really two points $D$ and $D'$ that are possible, although one is outside the triangle; they give corresponding points $E$ and $E'$. The circles $BEM$ and $BE'M$ must then actually coincide since they are both alleged to be tangent to line $AB$. See the figure below.




One can already prove using angle chasing that $\overline{AB}$ is tangent to $(BEE')$. So the point of the problem is to show that $M$ lies on this circle too. However, from looking at the diagram, one may realize that in fact it seems

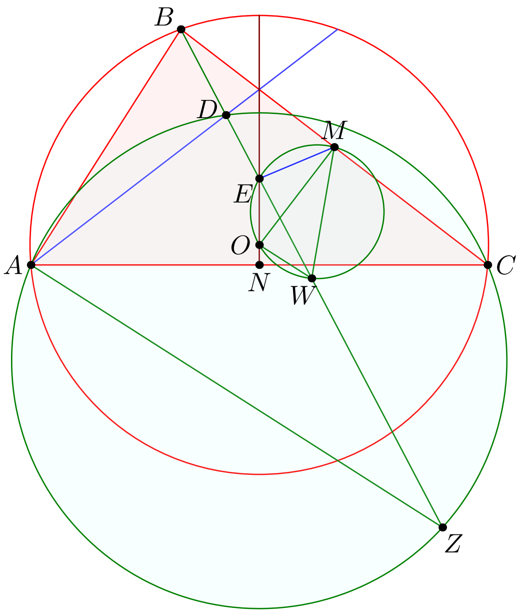
$$\triangle MEE' \overset{+}{\sim} \triangle CDD'$$

is going to be true from just those marked in the figure (and this would certainly imply the desired concyclic conclusion). Since $M$ is a midpoint, it makes sense to dilate $\triangle EME'$ from $B$ by a factor of 2 to get $\triangle FCF'$ so that the desired similarity is actually a spiral similarity at $C$. Then the spiral similarity lemma says that the desired similarity is equivalent to requiring $\overline{DD'} \cap \overline{FF'} = Q$ to lie on both $(CDF)$ and $(CD'F')$. Hence the key construction and claim from the solution are both discovered naturally, and we find the solution above. (The points $D'$, $E'$, $F'$ can then be deleted to hide the motivation.)

**Another short solution.** Let $Z$ be on line $BDE$ such that $\angle BAZ = 90^\circ$. This lets us interpret the angle condition as follows:

**Claim.** Points $A$, $D$, $Z$, $C$ are cyclic.

*Proof.* Because $\measuredangle ZAC = 90^\circ - A = 180^\circ - \measuredangle CDB = \measuredangle ZDC$. $\square$




Define $W$ as the midpoint of $\overline{BZ}$, so $\overline{MW} \parallel \overline{CZ}$. And let $O$ denote the center of $(ABC)$.

**Claim.** Points $M$, $E$, $O$, $W$ are cyclic.

*Proof.* Note that

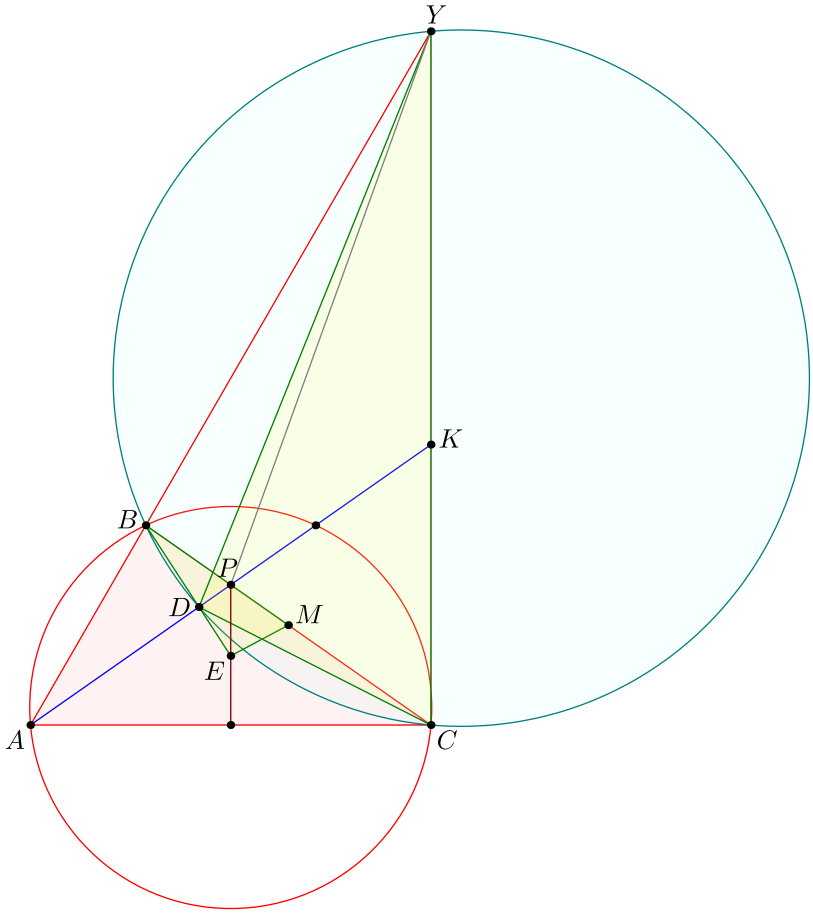
$$\begin{aligned}\measuredangle MOE &= \measuredangle(\overline{OM}, \overline{BC}) + \measuredangle(\overline{BC}, \overline{AC}) + \measuredangle(\overline{AC}, \overline{OE}) \\ &= 90^\circ + \measuredangle BCA + 90^\circ \\ &= \measuredangle BCA = \measuredangle CAD = \measuredangle CZD = \measuredangle MWD = \measuredangle MWE. \end{aligned}$$ □

To finish, note

$$\begin{aligned}\measuredangle MEB = \measuredangle MEW &= \measuredangle MOW \\ &= \measuredangle(\overline{MO}, \overline{BC}) + \measuredangle(\overline{BC}, \overline{AB}) + \measuredangle(\overline{AB}, \overline{OW}) \\ &= 90^\circ + \measuredangle CBA + 90^\circ = \measuredangle CBA = \measuredangle MBA.\end{aligned}$$

This implies the desired tangency.

**A Menelaus-based approach (Kevin Ren).** Let $P$ be on $\overline{BC}$ with $AP = PC$. Let $Y$ be the point on line $AB$ such that $\angle ACY = 90^\circ$; as $\angle AYC = 90^\circ - A$ it follows $BDYC$ is cyclic. Let $K = \overline{AP} \cap \overline{CY}$, so $\triangle ACK$ is a right triangle with $P$ the midpoint of its hypotenuse.




**Claim.** Triangles $BPE$ and $DYK$ are similar.

*Proof.* We have $\measuredangle MPE = \measuredangle CPE = \measuredangle KCP = \measuredangle PKC$ and $\measuredangle EBP = \measuredangle DBC = \measuredangle DYC = \measuredangle DYK$. □

**Claim.** Triangles $BEM$ and $YDC$ are similar.

*Proof.* By Menelaus $\triangle PCK$ with respect to collinear points $A$, $B$, $Y$ that

$$\frac{BP}{BC}\frac{YC}{YK}\frac{AK}{AP} = 1.$$

Since $AK/AP = 2$ (note that $P$ is the midpoint of the hypotenuse of right triangle $ACK$) and $BC = 2BM$, this simplifies to

$$\frac{BP}{BM} = \frac{YK}{YC}.$$

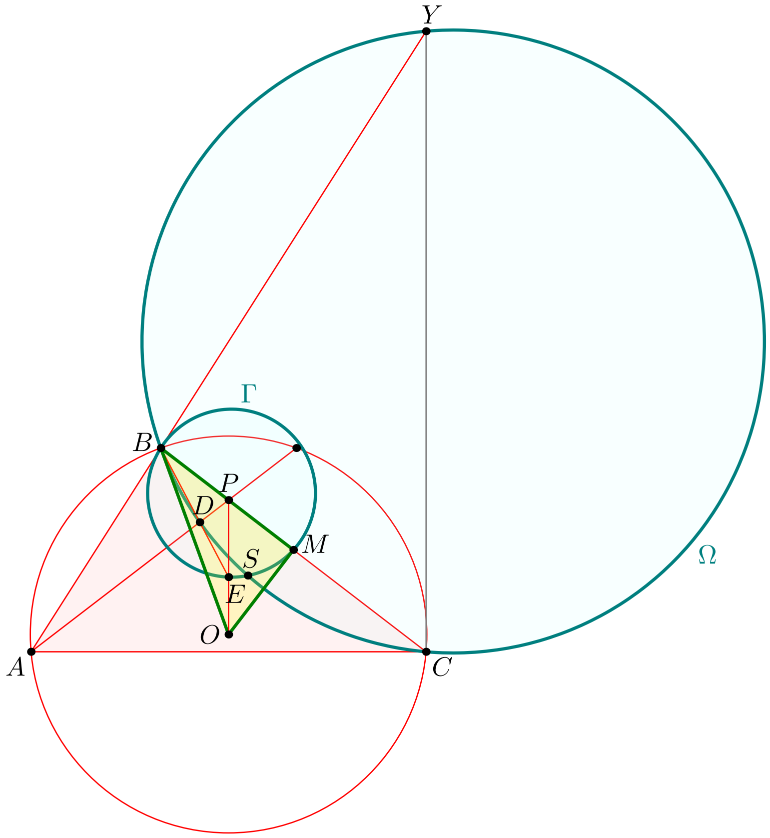
□

To finish, note that

$$\measuredangle DBA = \measuredangle DBY = \measuredangle DCY = \measuredangle BME$$

implying the desired tangency.

**A spiral similarity approach (Hans Yu).** As in the previous solution, let $Y$ be the point on line $AB$ such that $\angle ACY = 90°$; so $BDYC$ is cyclic. Let $\Gamma$ be the circle through $B$ and $M$ tangent to $\overline{AB}$, and let $\Omega := (BCYD)$. We need to show $E \in \Gamma$.




Denote by $S$ the second intersection of $\Gamma$ and $\Omega$. The main idea behind is to consider the spiral similarity

$$\Psi : \Omega \to \Gamma \qquad C \mapsto M \text{ and } Y \mapsto B$$

centered at $S$ (due to the spiral similarity lemma), and show that $\Psi(D) = E$. The spiral similarity lemma already promises $\Psi(D)$ lies on line $BD$.

**Claim.** We have $\Psi(A) = O$, the circumcenter of $ABC$.

*Proof.* Note $\triangle OBM \overset{+}{\sim} \triangle AYC$; both are right triangles with $\measuredangle BAC = \measuredangle BOM$. □

**Claim.** $\Psi$ maps line $AD$ to line $OP$.

*Proof.* If we let $P$ be on $\overline{BC}$ with $AP = PC$ as before,

$$\measuredangle(\overline{AD}, \overline{OP}) = \measuredangle APO = \measuredangle OPC = \measuredangle YCP = \measuredangle(\overline{YC}, \overline{BM}).$$

As $\Psi$ maps line $YC$ to line $BM$ and $\Psi(A) = O$, we're done. □

Hence $\Psi(D)$ should not only lie on $BD$ but also line $OP$. This proves $\Psi(D) = E$, so $E \in \Gamma$ as needed.