USAMO 2024/5

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Twitch Solves ISL

Episode 141

Problem

Point D is selected inside acute triangle ABC so that $\angle DAC = \angle ACB$ and $\angle BDC = 90^\circ + \angle BAC$. Point E is chosen on ray BD so that $AE = EC$. Let M be the midpoint of BC. Show that line AB is tangent to the circumcircle of triangle BEM.

Video

https://youtu.be/7M-b5zXXpaY

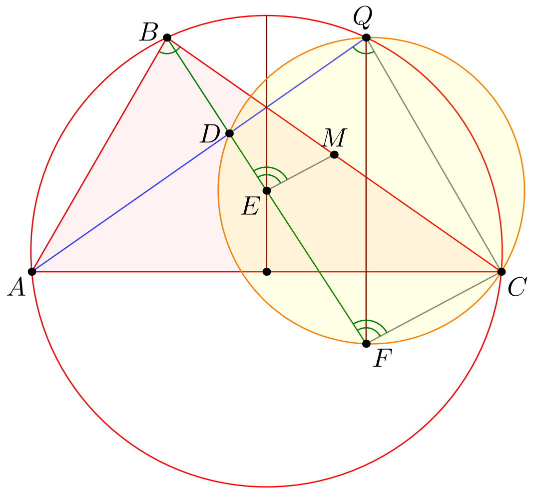
External Link

https://aops.com/community/p30227196

Solution

This problem has several approaches and we showcase a collection of them.

The author's original solution. Complete isosceles trapezoid $ABQC$ (so $D \in \overline{AQ}$). Reflect B across E to point F.



Claim. We have $DQCF$ is cyclic.

Proof. Since $EA = EC$, we have $\overline{QF} \perp \overline{AC}$ as line QF is the image of the perpendicular bisector of $\overline{AC}$ under a homothety from B with scale factor 2. Then

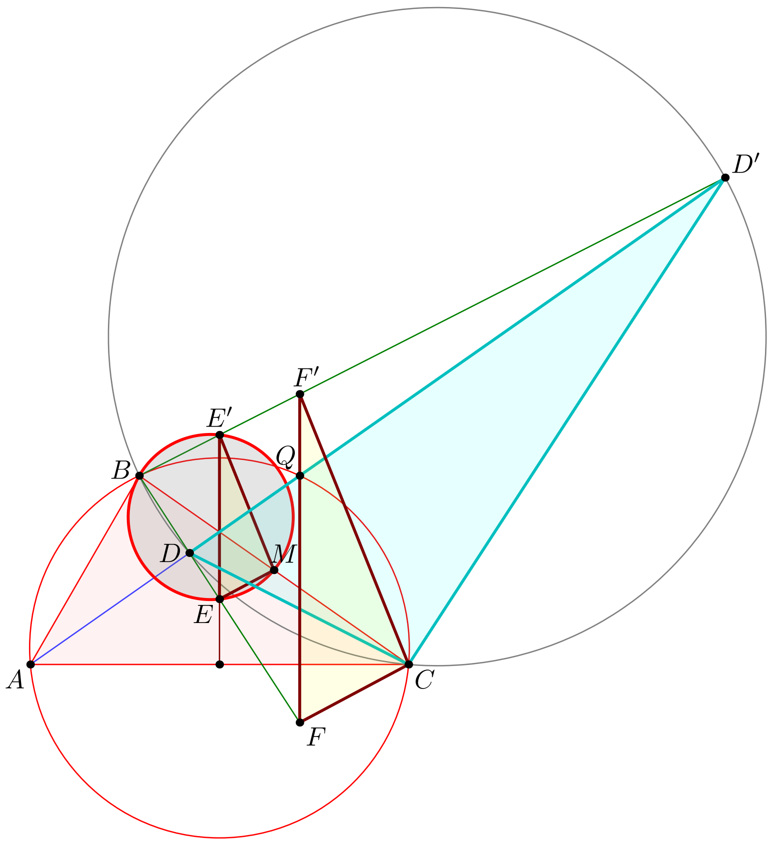
$$\begin{aligned} \measuredangle FDC &= -\measuredangle CDB = 180^\circ - (90^\circ + \measuredangle CAB) = 90^\circ - \measuredangle CAB \\ &= 90^\circ - \measuredangle QCA = \measuredangle FQC. \end{aligned}$$

□

To conclude, note that

$$\measuredangle BEM = \measuredangle BFC = \measuredangle DFC = \measuredangle DQC = \measuredangle AQC = \measuredangle ABC = \measuredangle ABM.$$

Remark (Motivation)**.** Here is one possible way to come up with the construction of point F (at least this is what led Evan to find it). If one directs all the angles in the obvious way, there are really two points D and D' that are possible, although one is outside the triangle; they give corresponding points E and E'. The circles BEM and $BE'M$ must then actually coincide since they are both alleged to be tangent to line AB. See the figure below.



One can already prove using angle chasing that $\overline{AB}$ is tangent to (BEE'). So the point of the problem is to show that M lies on this circle too. However, from looking at the diagram, one may realize that in fact it seems

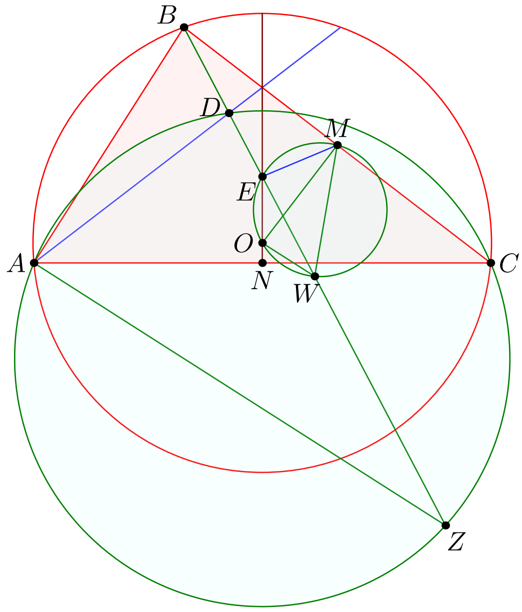
$$\triangle MEE' \overset{+}{\sim} \triangle CDD'$$

is going to be true from just those marked in the figure (and this would certainly imply the desired concyclic conclusion). Since M is a midpoint, it makes sense to dilate $\triangle EME'$ from B by a factor of 2 to get $\triangle FCF'$ so that the desired similarity is actually a spiral similarity at C. Then the spiral similarity lemma says that the desired similarity is equivalent to requiring $\overline{DD'} \cap \overline{FF'} = Q$ to lie on both (CDF) and $(CD'F')$. Hence the key construction and claim from the solution are both discovered naturally, and we find the solution above. (The points D', E', F' can then be deleted to hide the motivation.)

Another short solution. Let Z be on line BDE such that $\angle BAZ = 90°$. This lets us interpret the angle condition as follows:

Claim. Points A, D, Z, C are cyclic.

Proof. Because $\measuredangle ZAC = 90° - A = 180° - \measuredangle CDB = \measuredangle ZDC$. □



Define W as the midpoint of $\overline{BZ}$, so $\overline{MW} \parallel \overline{CZ}$. And let O denote the center of (ABC).

Claim. Points M, E, O, W are cyclic.

Proof. Note that

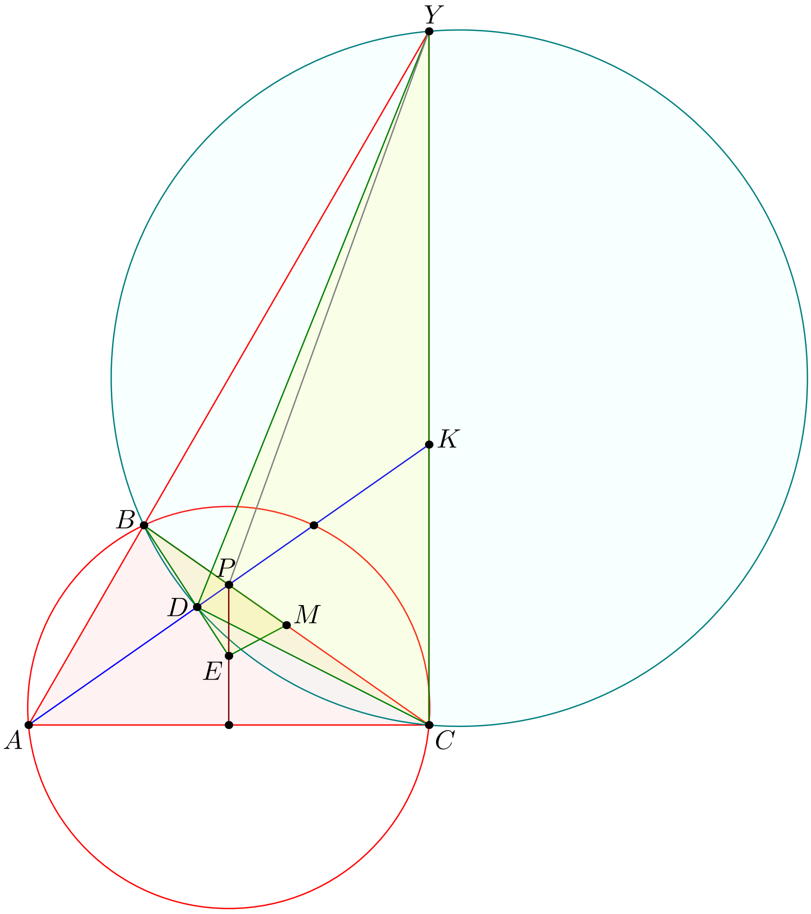
$$\begin{aligned}\measuredangle MOE &= \measuredangle(\overline{OM}, \overline{BC}) + \measuredangle(\overline{BC}, \overline{AC}) + \measuredangle(\overline{AC}, \overline{OE}) \\ &= 90^\circ + \measuredangle BCA + 90^\circ \\ &= \measuredangle BCA = \measuredangle CAD = \measuredangle CZD = \measuredangle MWD = \measuredangle MWE. \qquad \square\end{aligned}$$

To finish, note

$$\begin{aligned}\measuredangle MEB = \measuredangle MEW &= \measuredangle MOW \\ &= \measuredangle(\overline{MO}, \overline{BC}) + \measuredangle(\overline{BC}, \overline{AB}) + \measuredangle(\overline{AB}, \overline{OW}) \\ &= 90^\circ + \measuredangle CBA + 90^\circ = \measuredangle CBA = \measuredangle MBA.\end{aligned}$$

This implies the desired tangency.

A Menelaus-based approach (Kevin Ren). Let P be on $\overline{BC}$ with $AP = PC$. Let Y be the point on line AB such that $\angle ACY = 90^\circ$; as $\angle AYC = 90^\circ - A$ it follows $BDYC$ is cyclic. Let $K = \overline{AP} \cap \overline{CY}$, so $\triangle ACK$ is a right triangle with P the midpoint of its hypotenuse.



Claim. Triangles BPE and DYK are similar.

Proof. We have $\measuredangle MPE = \measuredangle CPE = \measuredangle KCP = \measuredangle PKC$ and $\measuredangle EBP = \measuredangle DBC = \measuredangle DYC = \measuredangle DYK$. □

Claim. Triangles BEM and YDC are similar.

Proof. By Menelaus $\triangle PCK$ with respect to collinear points A, B, Y that

$$\frac{BP}{BC}\frac{YC}{YK}\frac{AK}{AP} = 1.$$

Since $AK/AP = 2$ (note that P is the midpoint of the hypotenuse of right triangle ACK) and $BC = 2BM$, this simplifies to

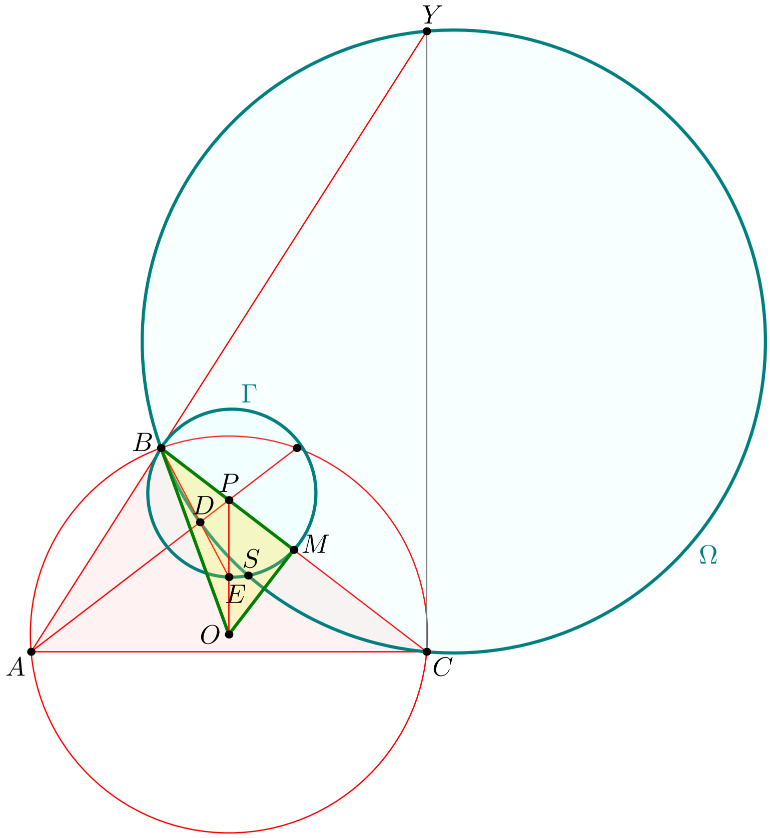
$$\frac{BP}{BM} = \frac{YK}{YC}.$$ □

To finish, note that

$$\measuredangle DBA = \measuredangle DBY = \measuredangle DCY = \measuredangle BME$$

implying the desired tangency.

A spiral similarity approach (Hans Yu). As in the previous solution, let Y be the point on line AB such that $\angle ACY = 90°$; so $BDYC$ is cyclic. Let Γ be the circle through B and M tangent to $\overline{AB}$, and let $\Omega := (BCYD)$. We need to show $E \in \Gamma$.



Denote by S the second intersection of Γ and Ω. The main idea behind is to consider the spiral similarity

$$\Psi : \Omega \to \Gamma \qquad C \mapsto M \text{ and } Y \mapsto B$$

centered at S (due to the spiral similarity lemma), and show that $\Psi(D) = E$. The spiral similarity lemma already promises $\Psi(D)$ lies on line BD.

Claim. We have $\Psi(A) = O$, the circumcenter of ABC.

Proof. Note $\triangle OBM \overset{+}{\sim} \triangle AYC$; both are right triangles with $\measuredangle BAC = \measuredangle BOM$. □

Claim. Ψ maps line AD to line OP.

Proof. If we let P be on $\overline{BC}$ with $AP = PC$ as before,

$$\measuredangle(\overline{AD}, \overline{OP}) = \measuredangle APO = \measuredangle OPC = \measuredangle YCP = \measuredangle(\overline{YC}, \overline{BM}).$$

As Ψ maps line YC to line BM and $\Psi(A) = O$, we're done. □

Hence $\Psi(D)$ should not only lie on BD but also line OP. This proves $\Psi(D) = E$, so $E \in \Gamma$ as needed.