JMO 2024/2

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Twitch Solves ISL

Episode 141

Problem

Let m and n be positive integers. Let S be the set of lattice points (x, y) with $1 \le x \le 2m$ and $1 \le y \le 2n$. A configuration of mn axis-parallel rectangles is called *happy* if each point of S is the vertex of exactly one rectangle. Prove that the number of happy configurations is odd.

Video

https://youtu.be/eZe8tDDSx70

External Link

https://aops.com/community/p30216444

Solution

There are several possible approaches to the problem; most of them involve pairing some of the happy configurations in various ways, leaving only a few configurations which remain fixed. We present the original proposer's solution and Evan's more complicated one.

Original proposer's solution. To this end, let's denote by $f(2m, 2n)$ the number of happy configurations for a $2m \times 2n$ grid of lattice points (not necessarily equally spaced — this doesn't change the count). We already have the following easy case.

Claim. We have $f(2, 2n) = (2n-1)!! = (2n-1) \cdot (2n-3) \cdot \dots \cdot 3 \cdot 1$.

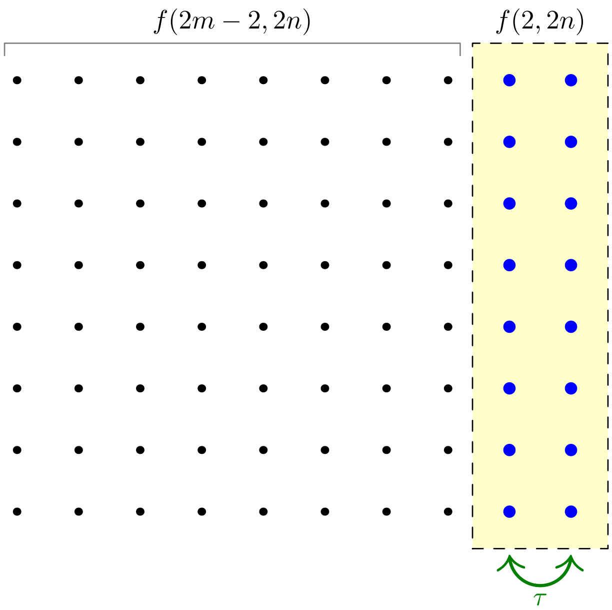
Proof. The top row is the top edge of some rectangle and there are $2n-1$ choices for the bottom edge of that rectangle. It then follows $f(2, 2n) = (2n-1) \cdot f(2, 2n-2)$ and the conclusion follows by induction on n, with $f(2, 2) = 1$. □

We will prove that:

Claim. Assume $m, n \geq 1$. When $f(2m, 2n) \equiv f(2m-2, 2n) \pmod 2$.

Proof. Given a happy configuration $\mathcal{C}$, let $\tau(\mathcal{C})$ be the happy configuration obtained by swapping the last two columns. Obviously $\tau(\tau(\mathcal{C})) = \mathcal{C}$ for every happy $\mathcal{C}$. So in general, we can consider two different kinds of configurations $\mathcal{C}$, those for which $\tau(\mathcal{C}) \neq \mathcal{C}$, so we get pairs $\{\mathcal{C}, \tau(\mathcal{C})\}$, and those with $\tau(\mathcal{C}) = \mathcal{C}$.

Now configurations fixed by τ can be described readily: this occurs if and only if the last two columns are self-contained, meaning every rectangle with a vertex in these columns is completely contained in these two columns.



Hence it follows that

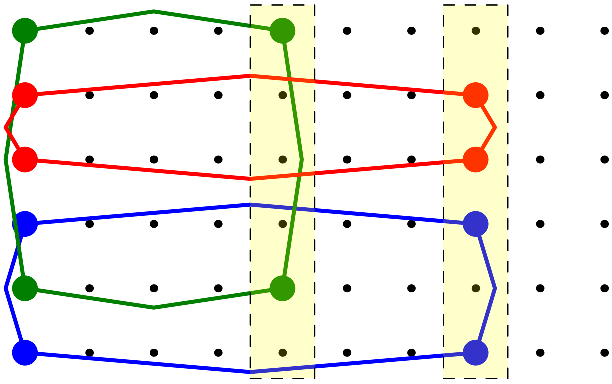
$$f(2m, 2n) = 2(\text{number of pairs}) + f(2m-2, 2n) \cdot f(2, 2n).$$

Taking modulo 2 gives the result. □

By the same token $f(2m, 2n) \equiv f(2m, 2n-2) \pmod 2$. So all f-values have the same parity, and from $f(2, 2) = 1$ we're done.

Remark. There are many variations of the solution using different kinds of τ. The solution with τ swapping two rows seems to be the simplest.

Evan's permutation-based solution. Retain the notation $f(2m, 2n)$ from before. Given a happy configuration, consider all the rectangles whose left edge is in the first column. Highlight every column containing the right edge of such a rectangle. For example, in the figure below, there are two highlighted columns. (The rectangles are drawn crooked so one can tell them apart.)



We organize happy configurations based on the set of highlighted columns. Specifically, define the relation $\sim$ on configurations by saying that $\mathcal{C} \sim \mathcal{C}'$ if they differ by any permutation of the highlighted columns. This is an equivalence relation. And in general, if there are k highlighted columns, its equivalence class under $\sim$ has $k!$ elements.

Then

Claim. $f(2m, 2n)$ has the same parity as the number of happy configurations with *exactly* one highlighted column.

Proof. Since $k!$ is even for all $k \geq 2$, but odd when $k = 1$. $\square$

There are $2m-1$ ways to pick a single highlighted column, and then $f(2, 2n) = (2n-1)!!$ ways to use the left column and highlighted column. So the count in the claim is exactly given by

$$(2m-1) \cdot (2n-1)!! f(2m-2, 2n).$$

This implies $f(2m, 2n) \equiv f(2m-2, 2n) \pmod 2$ and proceeding by induction as before solves the problem.