JMO 2024/1

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Twitch Solves ISL

Episode 141

Problem

Let $ABCD$ be a cyclic quadrilateral with $AB = 7$ and $CD = 8$. Points P and Q are selected on line segment AB so that $AP = BQ = 3$. Points R and S are selected on line segment CD so that $CR = DS = 2$. Prove that $PQRS$ is a cyclic quadrilateral.

Video

https://youtu.be/N3WG9AY9HYY

External Link

https://aops.com/community/p30216434

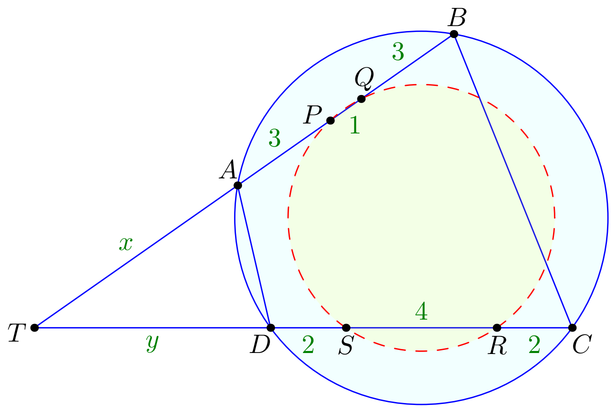
Solution

Here are two possible approaches.

The one-liner. The four points P, Q, R, S have equal power -12 with respect to $(ABCD)$. So in fact they're on a circle concentric with $(ABCD)$.

The external power solution. We distinguish between two cases.

Case where AB and CD are not parallel. We let lines AB and CD meet at T. Without loss of generality, A lies between B and T and D lies between C and T. Let $x = TA$ and $y = TD$, as shown below.



By power of a point,

$$ABCD \text{ cyclic} \iff x(x+7) = y(y+8)$$
$$PQRS \text{ cyclic} \iff (x+3)(x+4) = (y+2)(y+6).$$

However, the latter equation is just the former with 12 added to both sides. (That is, $(x+3)(x+4) = x(x+7)+12$ while $(y+2)(y+6) = y(y+8)+12$.) So the conclusion is immediate.

Case where AB and CD are parallel. In that case $ABCD$ is an isosceles trapezoid. Then the entire picture is symmetric around the common perpendicular bisector of the lines AB and CD. Now $PQRS$ is also an isosceles trapezoid, so it's cyclic too.