

JMO 2024/1

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TWITCH SOLVES ISL

Episode 141

Problem

Let $ABCD$ be a cyclic quadrilateral with $AB = 7$ and $CD = 8$. Points P and Q are selected on line segment AB so that $AP = BQ = 3$. Points R and S are selected on line segment CD so that $CR = DS = 2$. Prove that $PQRS$ is a cyclic quadrilateral.

Video

<https://youtu.be/N3WG9AY9HYY>

External Link

<https://aops.com/community/p30216434>

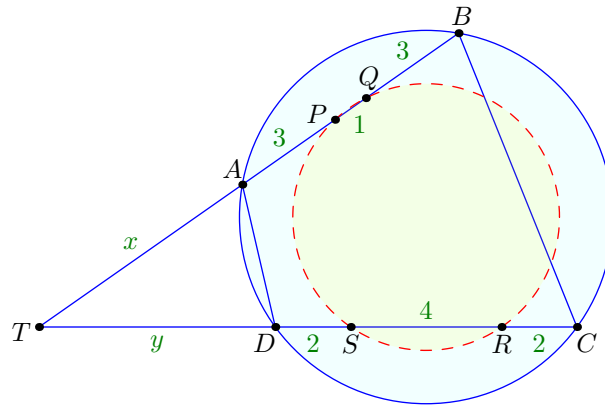
Solution

Here are three possible approaches.

The one-liner. The four points P, Q, R, S have equal power -12 with respect to $(ABCD)$. So in fact they're on a circle concentric with $(ABCD)$.

The external power solution. We distinguish between two cases.

Case where AB and CD are not parallel. We let lines AB and CD meet at T . Without loss of generality, A lies between B and T and D lies between C and T . Let $x = TA$ and $y = TD$, as shown below.

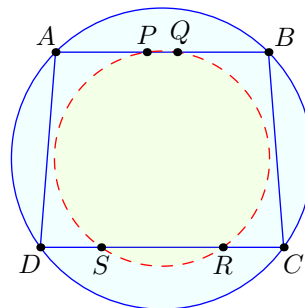


By power of a point,

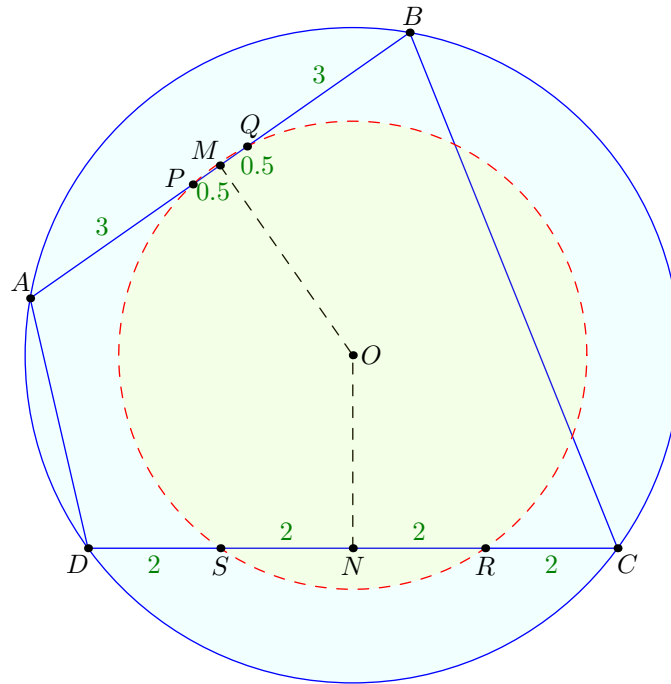
$$\begin{aligned}
 ABCD \text{ cyclic} &\iff x(x+7) = y(y+8) \\
 PQRS \text{ cyclic} &\iff (x+3)(x+4) = (y+2)(y+6).
 \end{aligned}$$

However, the latter equation is just the former with 12 added to both sides. (That is, $(x+3)(x+4) = x(x+7) + 12$ while $(y+2)(y+6) = y(y+8) + 12$.) So the conclusion is immediate.

Case where AB and CD are parallel. In that case $ABCD$ is an isosceles trapezoid. Then the entire picture is symmetric around the common perpendicular bisector of the lines AB and CD . Now $PQRS$ is also an isosceles trapezoid, so it's cyclic too.



The Pythagorean bash. Let ρ and O denote the radius and center of the circle circumscribing $ABCD$. We will show that in fact, P, Q, R, S lie on a circle centered at O , i.e. that the lengths PO, QO, RO, SO are all equal.



Our proof will use repeated applications of the Pythagorean theorem. Let M denote the midpoint of \overline{PQ} , which is also the midpoint of \overline{AB} , as $AM = MB = 3.5$ and $PM = QM = 0.5$. Assuming M is distinct from O , it follows \overline{OM} is the perpendicular bisector of \overline{AB} , and so the Pythagorean theorem we get

$$\begin{aligned} PO^2 &= PM^2 + MO^2 = PM^2 + (AO^2 - AM^2) = 0.5^2 + \rho^2 - 3.5^2 \\ QO^2 &= QM^2 + MO^2 = QM^2 + (BO^2 - BM^2) = 0.5^2 + \rho^2 - 3.5^2. \end{aligned}$$

And in the case $O = M$, the same equations are valid too.

Similarly let N denote the midpoint of \overline{RS} , which is also the midpoint of \overline{CD} , as $CN = DN = 4$ and $RN = SN = 2$. Repeating the same calculation gives

$$\begin{aligned} RO^2 &= RN^2 + NO^2 = RN^2 + (CO^2 - CN^2) = 2^2 + \rho^2 - 4^2 \\ SO^2 &= SN^2 + NO^2 = SN^2 + (DO^2 - DN^2) = 2^2 + \rho^2 - 4^2. \end{aligned}$$

From this it follows that

$$PO^2 = QO^2 = RO^2 = SO^2 = \rho^2 - 12.$$

Hence the problem is solved.