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Twitch Solves ISL

Episode 139

Problem

Positive integers a, b, c, d satisfy the conditions

$$\begin{aligned} a+b+c+d &= (a-b)^2 \\ ab &= cd. \end{aligned}$$

Show that $4c+1$ is a square.

Video

https://youtu.be/NCPmljDOITk

External Link

https://aops.com/community/p27901399

Solution

The product condition implies that we can choose integers $u, v, x, y > 0$ with

$$a = ux, \qquad b = vy, \qquad c = vx, \qquad d = uy.$$

In that case we can write

$$\begin{aligned} a + b + c + d &= (u + v)(x + y) \\ (a - b)^2 &= (ux - vy)^2. \end{aligned}$$

Define

$$\begin{aligned} \alpha &:= u + v \\ \beta &:= x + y. \end{aligned}$$

Then the given equation becomes

$$(x\alpha - v\beta)^2 = \alpha\beta.$$

This lets us set $\alpha = \gcd(\alpha, \beta)m^2$, $\beta = \gcd(\alpha, \beta)n^2$ for coprime $m, n > 0$. This implies that

$$xm^2 - vn^2 = \pm mn \implies x\left(\frac{m}{n}\right)^2 \mp \frac{m}{n} - v = 0.$$

So in particular, the quadratic equation

$$x\theta^2 \mp \theta - v = 0$$

in the variable θ has a rational solution $\frac{m}{n}$. Therefore the discriminant

$$(\mp 1)^2 - 4vx = 1 + 4vx = 4c + 1$$

is a perfect square, as desired.

Remark. For any positive integer k, an example of a satisfying quadruple is

$$(a, b, c, d) = \left(k^2, (k + 1)^2, k(k + 1), k(k + 1)\right)$$

which corresponds to

$$\begin{aligned} (u, v, x, y) &= (k, k + 1, k, k + 1) \\ \alpha = \beta &= 2k + 1 \\ m = n &= 1. \end{aligned}$$

Remark. The problem is actually false if the hypothesis $a, b, c, d > 0$ is dropped, but only in the situation where the variables α and β could equal zero. For example, this happens if $(a, b, c, d) = (1, 1, -1, -1)$.