China TST Quiz 2007/1/3

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Twitch Solves ISL

Episode 139

Problem

Fix an integer $n \geq 1$. Prove there exists exactly one polynomial $P(x)$ of degree n with real coefficients, such that $P(0) = 1$ and $(x+1)(P(x))^2 - 1$ is odd.

Video

https://youtu.be/rmhfgp47avs

External Link

https://aops.com/community/p1365013

Solution

Suppose the polynomial P obeys

$$2 = (x+1)P(x)^2 + (-x+1)P(-x)^2.$$

Work in $\mathbb{R}[y, y^{-1}]$ and substitute

$$x = \frac{1}{2}\left(y + \frac{1}{y}\right).$$

Then the above equation can be rewritten as

$$\begin{aligned} 4 &= \left(y + \frac{1}{y} + 2\right) P(x)^2 - \left(y + \frac{1}{y} - 2\right) P(-x)^2 \\ \iff 4y &= ((y+1)P(x))^2 - ((y-1)P(-x))^2 \\ \iff 4y &= \Big((y+1)P(x) + (y-1)P(-x)\Big)\Big((y+1)P(x) - (y-1)P(-x)\Big). \end{aligned}$$

Using factorization in $\mathbb{R}[y, y^{-1}]$, we see there must exist an integer $d \ge 0$ and a nonzero real number ε such that we have identities

$$\begin{aligned} 2\varepsilon \cdot y^{d+1} &= (y+1)P(x) \pm (y-1)P(-x) \\ \frac{2}{\varepsilon} \cdot y^{-d} &= (y+1)P(x) \mp (y-1)P(-x) \end{aligned}$$

for some opposite choice of signs $\pm$. Replacing y with $1/y$, it follows $\varepsilon = \varepsilon^{-1} \implies \varepsilon = \pm 1$. So the system of equations is equivalent to

$$\begin{aligned} y^{d+1} + y^{-d} &= \varepsilon(y+1)P(x) \\ y^{d+1} - y^{-d} &= \pm\varepsilon(y-1)P(-x) \end{aligned}$$

Now, the first equation implies that

$$\begin{aligned} \varepsilon P(x) &= \frac{y^{d+1} + y^{-d}}{y+1} \\ &= \left(y^d + \frac{1}{y^d}\right) - \left(y^{d-1} + \frac{1}{y^{d-1}}\right) + \dots + (-1)^{d-1}\left(y + \frac{1}{y}\right) + (-1)^d. \end{aligned}$$

The theory of Chebyshev polynomials guarantees that $y^k + \frac{1}{y^k}$ is a degree-k polynomial in x for each $k \ge 0$. Hence we see there is a unique choice of polynomial P of degree n (corresponding to $d = n$) which makes the identity $P(x) = \varepsilon \frac{y^{d+1}+y^{-d}}{y+1}$ true, for each $\varepsilon \in \{\pm 1\}$. When we add the condition $P(0) = 1$ (set $y = i$ so $x = 0$), we find only one $\varepsilon \in \{\pm 1\}$ is valid.

Conversely, we have the implication

$$\varepsilon(y+1)P(x) = y^{d+1} + y^{-d} \implies \varepsilon(y-1)P(-x) = \pm(y^{d+1} - y^{-d})$$

which follows readily by making the substitution $y \mapsto iy$. So the choice of polynomial P we just mentioned will satisfy the entire system, ergo (reversing the logic above) will satisfy the desired condition. In other words, the unique possibility for P does indeed work.

Remark. The first few actual polynomials P are:

$$\begin{aligned}
P(x) &= 1 \\
P(x) &= 1 - 2x \\
P(x) &= 1 + 2x - 4x^2 \\
P(x) &= 1 - 4x - 4x^2 + 8x^3 \\
P(x) &= 1 + 4x - 12x^2 - 8x^3 + 16x^4.
\end{aligned}$$