XOOK 2023/3

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Twitch Solves ISL

Episode 138

Problem

A *pinwheel* is defined to be any set of three rays, emanating from the same point, with $120°$ angles between any two rays. Let each pinwheel divide the plane into three *sectors*.

Let $\mathcal{S}$ be any set of distinct $3n$ points in the plane. Is it always possible to place a pinwheel such that exactly n points of $\mathcal{S}$ lie in the interior of each sector?

Video

https://youtu.be/lNoAOiFX1ys

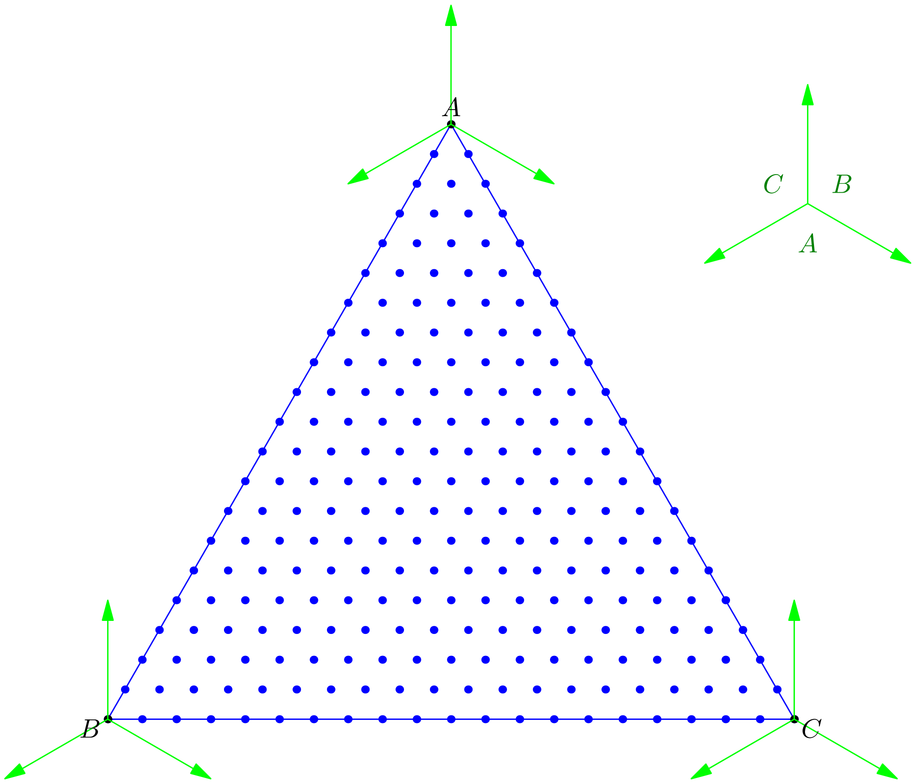
External Link

https://aops.com/community/p29571589

Solution

The answer is yes.

Rotate the plane such that no line through any two points of $\mathcal{S}$ has an argument whose argument is a rational number of degrees. Then contain all the points in a large upwards-facing equilateral triangle as shown below. Let N be a large positive integer and divide the triangle into a mesh of side length N. We are going to commit to taking a pinwheel whose rays point in the 90°, 210°, 330° directions centered at one of the points. By adjusting the value of N if needed (say, to a suitable large prime), we can also guarantee that pinwheels never pass through a point while on this lattice.



As we consider moving the pinwheel across consecutive points in the lattice, we get some number of points in the three regions A, B, C. These numbers change as we move the pinwheel; but if we pick N large enough, we can ensure that at most one point moves between regions as we perturb the pinwheel.

We now label every point in the lattice with the name of the region with the *most* points out of the $3n$ in it. Whenever there are ties, we pick the alphabetically earliest one.

Now by **Sperner lemma** there is a small equilateral triangle somewhere in this mesh whose three labels are different. Then the vertex with label A will work, because we see that moving it to the two neighbors would cause the label to change from A to B or C.