Mexico Q3

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Twitch Solves ISL

Episode 138

Problem

Let Γ_1 and Γ_2 be circles intersecting at points A and B. A line through A intersects Γ_1 and Γ_2 at C and D respectively. Let P be the intersection of the lines tangent to Γ_1 at A and C, and let Q be the intersection of the lines tangent to Γ_2 at A and D. Let X be the second intersection point of the circumcircles of BCP and BDQ, and let Y be the intersection of lines AB and PQ. Prove that C, D, X and Y are concyclic.

Video

https://youtu.be/c5NGYpfv8OQ

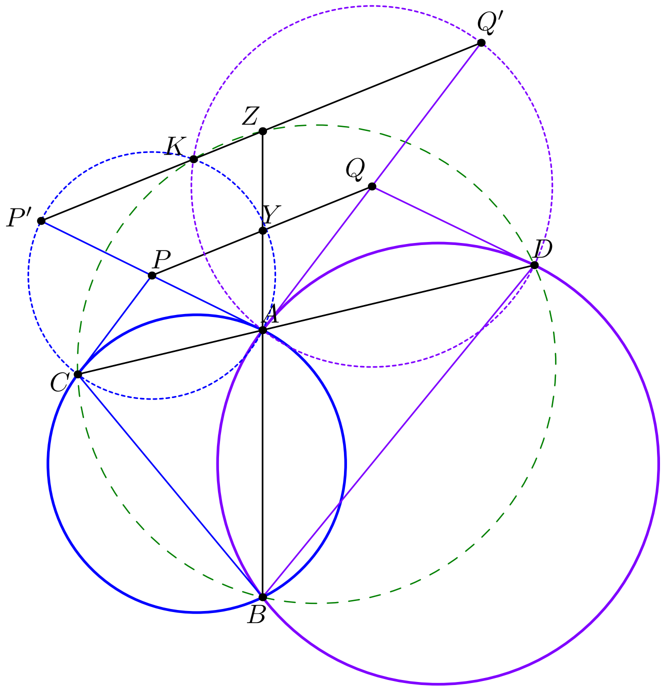
External Link

https://aops.com/community/p14889633

Solution

Ignore the point X for now and focus on getting control over Y. We start with the following claim:

Claim. Let Z be the reflection of A over Y. Then Z lies on (BCD).



Proof. Reflect A over P and Q to get P' and Q', collinear with Z. Then let K be the foot from A to $P'Q'$, which lies on the circles with diameter $\overline{AP'}$ and $\overline{AQ'}$ (that pass through C and D respectively). Now,

$$\measuredangle BCK = \measuredangle BCD + \measuredangle DCK = \measuredangle ZAP + \measuredangle AP'K = \measuredangle AZK = \measuredangle BZK$$

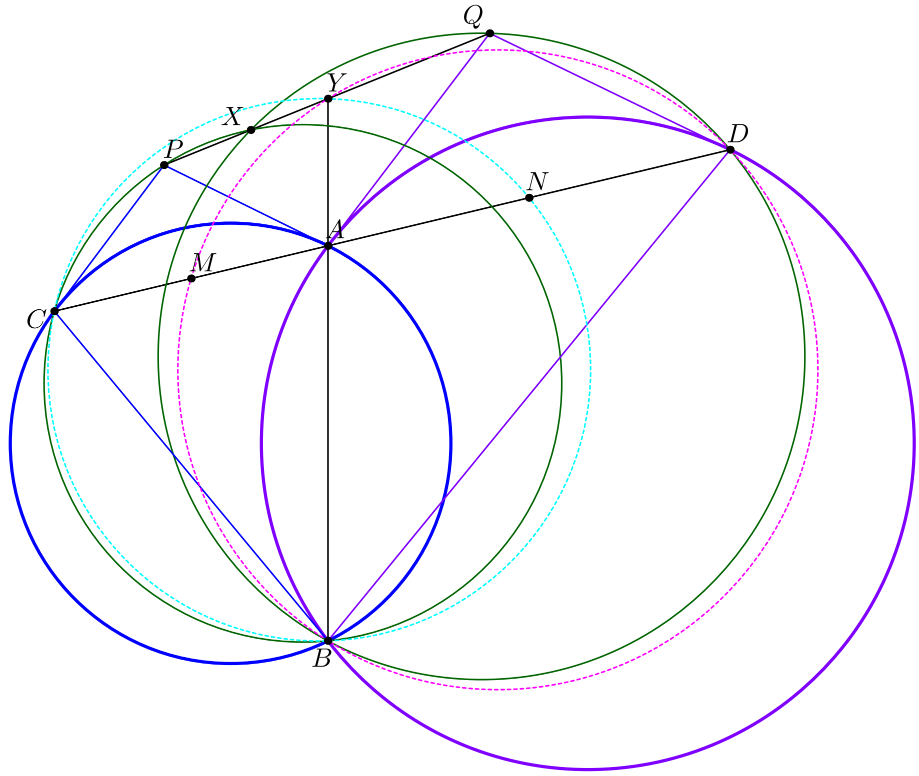
so C lies on (BZK). Then so does D, and we're done. □

We then trade in Z for the midpoints M and N of $\overline{CA}$ and $\overline{AD}$.

Claim. Point Y lies on both (CBN) and (DBM).

Proof. Follows by power of a point from A on the previous claim. □

Bring back point X now.



Now, for the main calculation, we have from symmedian theory that

$$\measuredangle CXB = \measuredangle CPB = \measuredangle BMA = \measuredangle BMD = \measuredangle BYD$$
$$\measuredangle BXD = \measuredangle BQD = \measuredangle ANB = \measuredangle CNB = \measuredangle CYB.$$

So summing gives $\measuredangle CXD = \measuredangle CYD$, as needed.

Remark. Note that X also lies on line PQ because $\measuredangle PXB = \measuredangle PCB = \measuredangle CAB = \measuredangle DAB = \measuredangle QDB = \measuredangle QXB$.