H3226403

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Twitch Solves ISL

Episode 138

Problem

Let ABC be a triangle with circumcircle Ω and orthocenter H. Let the midpoint of AH be N and let the reflections of N over $\overline{AC}$, $\overline{AB}$ be N_1, N_2, respectively. Let the line perpendicular to N_1N_2 through N_1 intersect AC at K and let the line perpendicular to N_1N_2 through N_2 intersect AB at L. The circumcircle of triangle AKL intersect Ω at A and X; let HX intersect Ω at X and Y. Show that $\overline{BY} \parallel \overline{HL}$ and $\overline{CY} \parallel \overline{HK}$.

Video

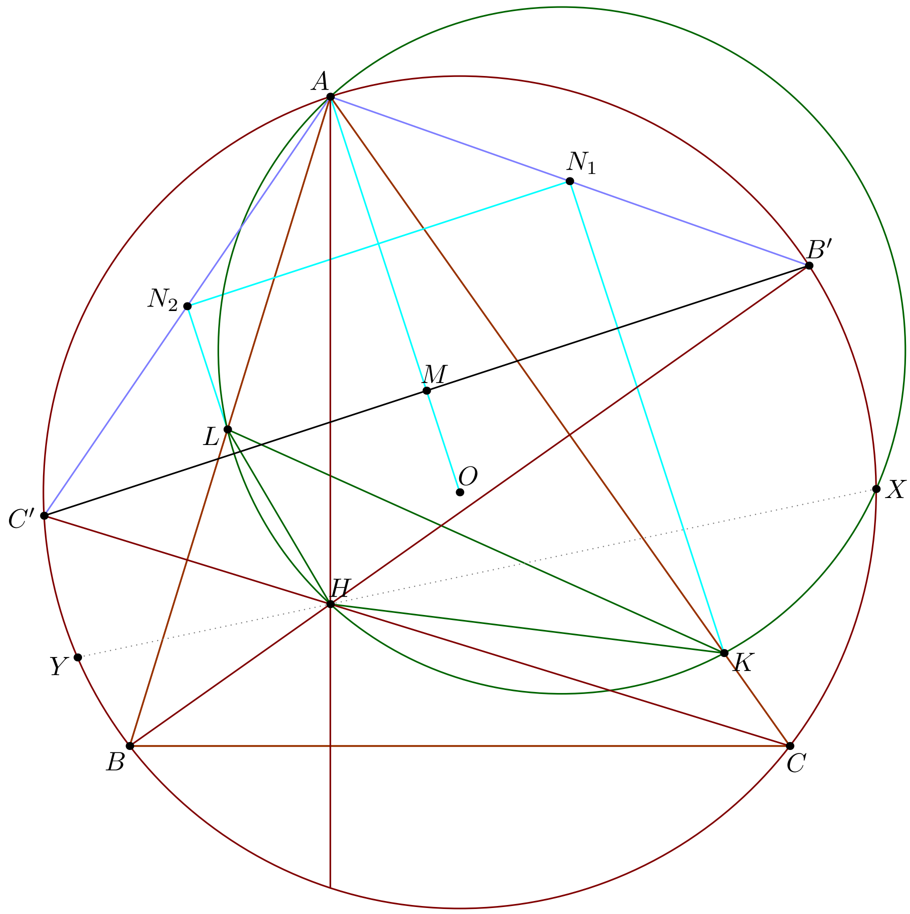
https://youtu.be/YTRMS2MG8P0

External Link

https://aops.com/community/p29539773

Solution

Let $\overline{BH}$ and $\overline{CH}$ meet Ω again at B' and C', respectively. It's known that $\overline{AO} \perp \overline{B'C'}$, and these two lines meet at the midpoint M of $\overline{B'C'}$. Ignore points X and Y for now.



Claim. Point L is the circumcenter of $\triangle MHC'$. Similarly, point K is the circumcenter of $\triangle MHB'$.

Proof. The perpendicular bisector of $\overline{HC'}$ is line MB while the perpendicular bisector of $\overline{C'M}$ is the line through N_2 perpendicular to $\overline{C'M} \parallel \overline{N_1N_2}$. So these two perpendicular bisectors indeed meet at L. $\square$

Claim. Points K, H, L, A are cyclic.

Proof. We see $\overline{KL}$ is the perpendicular bisector of $\overline{MH}$. Then, it follows that

$$\measuredangle MHL = 90^\circ - \measuredangle HC'M = 90^\circ - \measuredangle CC'B' = 90^\circ - \measuredangle CBB' = \measuredangle ACB.$$

Similarly, $\measuredangle KHM = \measuredangle CBA$. So $\measuredangle KHL = \measuredangle KHM + \measuredangle MHL = \measuredangle CAB = \measuredangle KAL$. $\square$

Remark. Alternatively, trig Ptolemy provides a straightforward way to prove the second claim without having noticed the circumcenters from before.

Remark. In fact, quadrilateral $ALHK$ is harmonic too. Indeed, $\overline{AO}$ is the median of $\triangle ALK$, because line $\overline{AO}$ is halfway between parallel lines $\overline{LN_2}$ and $\overline{N_1K}$, And lines $\overline{AO}$ and $\overline{AH}$ are isogonal in $\angle LAK$.

So actually M is the A-Humpty point of $\triangle AKL$, while N is the A-Dumpty point. However, we will not use this in what follows.

We finally turn our attention to X and Y which we previously ignored. To finish, we simply use Reim's theorem in the form

$$\measuredangle BYH = \measuredangle BYX = \measuredangle BAX = \measuredangle LAX = \measuredangle LHA.$$