EMCC 2023 T-14

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TWITCH SOLVES ISL

Episode 138

Problem

Equilateral hexagon $ABCDEF$ has opposite sides parallel, side length 3, and area 5. Find $AC \cdot BD \cdot CE \cdot DF \cdot EA \cdot FB$.

Video

https://youtu.be/FjKluBxpqRk

Solution

Construct parallelogram $COED$. Then $ABCO$ is also a parallelogram (because $\vec{B} - \vec{A} = \vec{D} - \vec{E} = \vec{C} - \vec{O}$), and similarly $AFEO$ is a parallelogram. So the point O thus the circumcenter of triangle ACE, because $AO = CO = EO = 3$, as shown below.
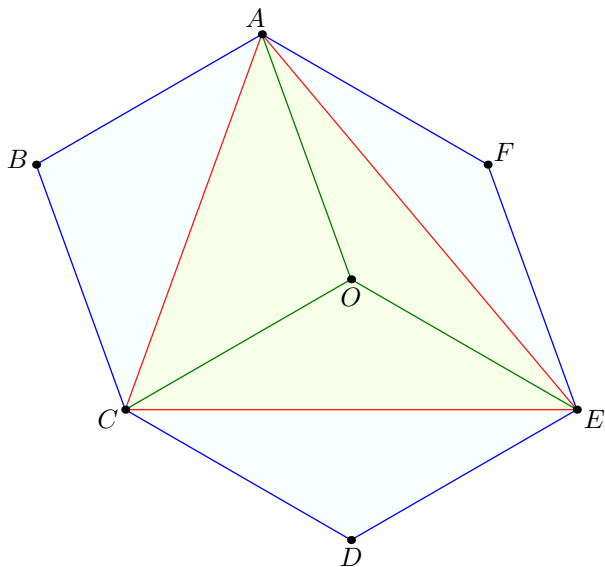
In particular, we have

$$\frac{AC \cdot CE \cdot EA}{4 \cdot 3} = \text{Area}(ACE) = \frac{\text{Area}(ABCDEF)}{2} = \frac{5}{2}$$

and hence

$$AC \cdot CE \cdot EA = 30.$$

Now $DF = AC$, $FB = CE$, $BD = EA$ so the answer is $30^2 = \boxed{900}$.