Twitch 137.3

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Twitch Solves ISL

Episode 137

Problem

For every positive integer n, there exists exactly one function $f_n\colon \{1, 2, \dots, n\} \to \mathbb{R}$ such that for all positive integers $k \le n$,

$$f_n(k) + f_n(2k) + \dots + f_n\left(k \left\lfloor \frac{n}{k} \right\rfloor\right) = \frac{1}{k} \left\lfloor \frac{n}{k} \right\rfloor.$$

Find $f_{1000}(1) - f_{999}(1)$.

Video

https://youtu.be/zDQbD1BOFCw

External Link

https://aops.com/community/p28878234

Solution

We solve the system of equations much more generally as follows:

Claim. Fix an integer $n \geq 1$. Consider the system of equations in n variables x_1, x_2, $\dots$, given by

$$\begin{aligned} c_1 &= x_1 + x_2 + x_3 + x_4 + \cdots \\ c_2 &= x_2 + x_4 + x_6 + x_8 + \cdots \\ c_3 &= x_3 + x_6 + x_9 + x_{12} + \cdots \\ &\vdots \\ c_n &= x_n \end{aligned}$$

where the k'th equation has $x_k + x_{2k} + \cdots + x_{k\lfloor n/k \rfloor}$ on the right-hand side. Then

$$x_1 = \sum_{k \leq n} \mu(k) c_k$$

where μ is the Möbius function.

Proof. If we substitute in the expressions for c_k on the right-hand side, the coefficient of x_m is $\sum_{k \mid m} \mu(k)$ which is known to be 1 for $m = 1$ and 0 otherwise. $\square$

Applying this directly here we get

$$\begin{aligned} f_{1000}(1) &= \sum_{k \leq 1000} \mu(k) \cdot \frac{1}{k} \cdot \left\lfloor \frac{1000}{k} \right\rfloor \\ f_{999}(1) &= \sum_{k \leq 999} \mu(k) \cdot \frac{1}{k} \cdot \left\lfloor \frac{999}{k} \right\rfloor . \end{aligned}$$

Note that

$$\left\lfloor \frac{1000}{k} \right\rfloor - \left\lfloor \frac{999}{k} \right\rfloor = \begin{cases} 1 & k \mid 1000 \\ 0 & k \nmid 1000. \end{cases}$$

So subtracting, the final answer is

$$\sum_{k \mid 1000} \frac{\mu(k)}{k} = 1 - \frac{1}{2} - \frac{1}{5} + \frac{1}{10} = \frac{2}{5}.$$