Twitch 137.2

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Twitch Solves ISL

Episode 137

Problem

In triangle ABC, let H and O denote the orthocenter and circumcenter, respectively. Define $D = \overline{CH} \cap \overline{AB}$, $E = \overline{AH} \cap \overline{CO}$, $F = \overline{AO} \cap \overline{BC}$, $P = \overline{FH} \cap \overline{CO}$, $Q = \overline{DE} \cap \overline{BO}$. Prove that $\angle BAP = \angle CAQ$.

Video

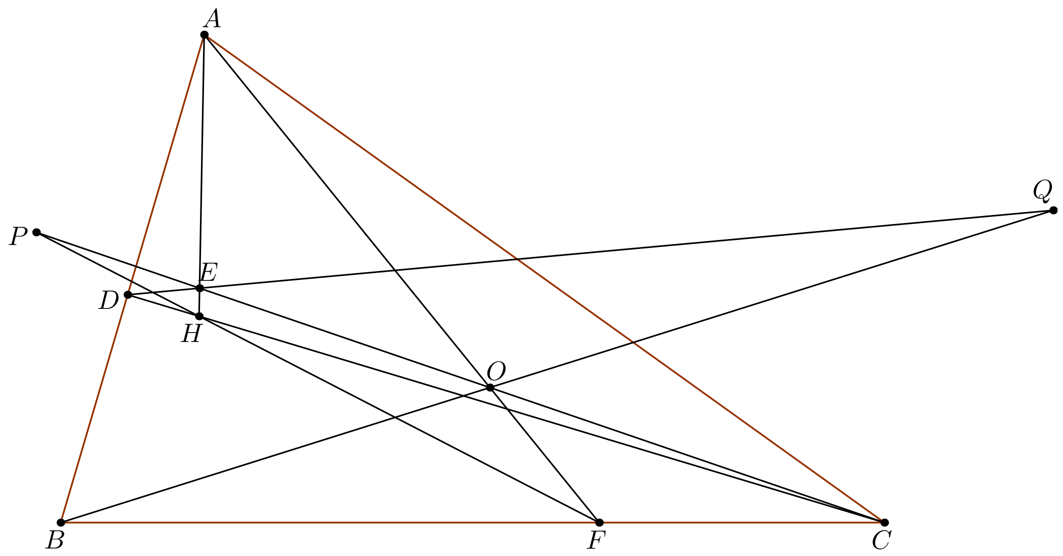
https://youtu.be/1VW8ETоKfWs

External Link

https://aops.com/community/p29310655

Solution

The problem is true for any pair of isogonal conjugates in place of (H, O). We use trilinear coordinates, with $H = (u : v : w)$ and $O = (1/u : 1/v : 1/w)$.



Compute the points:

$$\begin{aligned}
D &= (u : v : 0) \\
E &= (v^2 : uv : uw) \\
F &= (0 : w : v) \\
P &= (v : u : k) \implies \det(PHF) = 0 \\
Q &= (w : \ell : u) \implies \det(DEQ) = 0
\end{aligned}$$

We solve for k and ℓ in turn: for k we have

$$0 = \det \begin{bmatrix} v & u & k \\ u & v & w \\ 0 & w & v \end{bmatrix} = v(v^2 - w^2) + u(wk - uv)$$

hence

$$k = v \cdot \frac{w^2 - v^2 + u^2}{uw}.$$

Meanwhile for ℓ we have:

$$0 = \det \begin{bmatrix} u & v & 0 \\ v^2 & uv & uw \\ w & \ell & u \end{bmatrix} = u(uuv - v^3) + uw(vw - \ell u)$$

hence

$$\ell = v \cdot \frac{u^2 - v^2 + w^2}{uw}$$

We see that $k = \ell$. The isogonality $\angle BAP = \angle CAQ$ follows.